Quantifying Liquidity and Default Risks of Corporate Bonds over the Business Cycle*

Hui Chen Rui Cui Zhiguo He Konstantin Milbradt

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Abstract

We develop a structural credit risk model with time-varying macroeconomic risks and endogenous liquidity frictions. The model not only matches the average default probabilities, recovery rates, and average credit spreads for corporate bonds across different credit ratings, but also can account for bond liquidity measures including Bond-CDS spreads and bid-ask spreads across ratings. We propose a novel structural decomposition scheme of the credit spreads to capture the interaction between liquidity and default risk in corporate bond pricing. As an application, we use this framework to quantitatively evaluate the effects of liquidity-provision policies for the corporate bond market.

Keywords: Positive Liquidity-Default Feedback, Procyclical Liquidity, Rollover Risk, Search in Over-The-Counter Market, Endogenous Default, Structural Models

*Chen: MIT Sloan School of Management, and NBER; e-mail: huichen@mit.edu. Cui: Booth School of Business, University of Chicago; e-mail: rcui@chicagobooth.edu. He: Booth School of Business, University of Chicago, and NBER; e-mail: zhiguo.he@chicagobooth.edu. Milbradt: Kellogg School of Management, Northwestern University, and NBER; email: milbradt@northwestern.edu. We thank Mark Carey, Pierre Collin-Dufresne, David Lando, Lars Hansen, Mads Stenbo Nielsen, Vyacheslav Fos, and seminar participants at Chicago economic dynamics working group, ECB, Georgia Tech, Kellogg, Maryland, MIT Sloan, the NBER Summer Institute (risk of financial institution), Federal Reserve Board, AFA, and the USC Fixed Income Conference, University of Alberta, University of Calgary for helpful comments.

1. Introduction

It is well known that a significant part of the corporate bond pricing cannot be accounted for by default risk alone. For example, Longstaff, Mithal, and Neis (2005) estimate that the "non-default components" account for about 50% of the spreads between the yields of Aaa/Aa-rated corporate bonds and Treasuries and about 30% of the spreads for Baa-rated bonds. Furthermore, Longstaff, Mithal, and Neis (2005) find that the non-default components of credit spreads are strongly related to measures of bond liquidity, which is consistent with the evidence of illiquidity of the secondary corporate bond market (e.g., Edwards, Harris, and Piwowar (2007), Bao, Pan, and Wang (2011)), but weakly related to the differential state tax treatment on corporate bonds and Treasuries.

The literature on structural credit risk modeling has almost exclusively focused on the "default component" of credit spreads. The "credit spread puzzle," as defined by Huang and Huang (2012), refers to the finding that when calibrated to match the observed default rates and recovery rates, traditional structural models produce credit spreads for investment grade bonds that are significantly lower than those in the data. By introducing macroeconomic risks into the structural models, Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010) and Chen (2010) are able to explain the default components of the credit spreads for investment-grade bonds.[1] However, these models do not explain the sources of the non-default components in credit spreads, or their potential impact on credit risk.

In this paper, we build a tractable structural credit risk model that captures both the default and non-default components in corporate bond pricing. We introduce secondary over-the-counter market search frictions (a la Duffie, Gârleanu, and Pedersen (2005)) into a structural credit risk model with aggregate macroeconomic fluctuations (e.g., Chen (2010)). Rather than simply piecing together the default and non-default components of credit spreads,

[1] Chen (2010) relies on the estimates of Longstaff, Mithal, and Neis (2005) to obtain the default component of the credit spread for Baa rated bonds, while Bhamra, Kuehn, and Strebulaev (2010) focus on the difference between Baa and Aaa rated bonds. The difference of spreads between Baa and Aaa rated bonds presumably takes out the common liquidity component, which is a widely used practice in the literature. This treatment is accurate only if the liquidity components for both bonds are the same, which is at odds with existing literatures on liquidity of corporate bonds, e.g., Edwards, Harris, and Piwowar (2007), Bao, Pan, and Wang (2011).

our approach of jointly modeling default and liquidity risk over the business cycle is essential to capture the interactions between the two components. Such interactions give rise to the endogenous liquidity, which helps explain two general empirical patterns for the liquidity of corporate bonds: (1) corporate bonds with higher credit ratings tend to be more liquid; (2) corporate bonds are less liquid during economic downturns, especially for riskier bonds.[2] The interactions can also significantly raise the level of credit spreads and make them more volatile over the business cycle.

In our model which builds on He and Milbradt (2014), bond investors face the risk of uninsurable idiosyncratic liquidity shocks, which drive up their costs for holding the bonds. Market illiquidity arises endogenously because to sell their bonds, investors have to search for dealers to intermediate transactions with other investors not yet hit by liquidity shocks. The dealers set bid-ask spreads to capture part of trading surplus, and default risk affects the trading surplus and thus the liquidity discount of corporate bonds. The endogenous liquidity is further amplified by the endogenous default decision of the equity holders, as shown in Leland and Toft (1996) and emphasized by He and Xiong (2012). A default-liquidity spiral arises: when secondary market liquidity deteriorates, equity holders suffer heavier rollover losses in refinancing their maturing bonds and will consequently default earlier. This earlier default in turn worsens secondary bond market liquidity even further, and so on so forth.

Our model distinguishes from He and Milbradt (2014) in two important aspects. First, unlike He and Milbradt (2014) who take the holding cost due to liquidity shocks as a constant parameter, we consider a holding cost that increases with lower bond market prices and provide a micro-foundation based on the mechanism of collateralized borrowing. In this mechanism, investors hit by liquidity shocks raise cash either via the cheaper collateralized borrowing (with the bond being the collateral, subject to some haircuts) or the more expensive uncollateralized borrowing. A lower bond market price together with a higher haircut pushes investors toward the more expensive uncollateralized borrowing, translating to higher effective holding costs. Second, our paper introduces aggregate states into He and Milbradt (2014) by explicitly modeling the procyclical liquidity together with cyclical variations in the firm's cash

[2] See e.g., Edwards, Harris, and Piwowar (2007), Bao, Pan, and Wang (2011), Dick-Nielsen, Feldhütter, and Lando (2012), and Friewald, Jankowitsch, and Subrahmanyam (2012a).

flows and aggregate risk prices, which allows us the investigate the default-liquidity spiral across business cycles. Also, as the secondary market liquidity worsens (e.g., the meeting intensity with dealers goes down) in recessions when the marginal utility is high, our model gives rise to the liquidity risk premium in corporate bonds.

We follow the literature to set the pricing kernel parameters over two macroeconomic states (economic expansions and recessions) to fit key moments of asset prices that are outside the model. The parameters governing secondary bond market liquidity over macroeconomic states are either pre-fixed based on existing empirical studies and TRACE data (e.g., bond turnovers), or calibrated to match the moments that the model aims to explain (e.g., bid-ask spreads). We then apply our model to corporate bonds across four credit rating classes (Aaa/Aa, A, Baa, and Ba) and two different time-to-maturities (5 and 10 years). It is worth emphasizing that we only calibrate four free parameters, a much lower degree of freedom relative to the number of moments that our model tries to explain.

To evaluate model performance, we not only examine the cumulative default probabilities and credit spreads – two common measures that the previous literature on corporate bond pricing has focused on, but also consider two empirical measures of non-default risk for corporate bonds. The first measure is the Bond-CDS spreads, defined as the bond's credit spread minus the Credit Default Swap (CDS) spread on the same bond. This measure is motivated by Longstaff, Mithal, and Neis (2005) who argue that CDS contracts mostly price the default risk of bonds because of their more liquid secondary market. The second measure is the bond bid-ask spreads. These two measures crucially rely on secondary market illiquidity: in a model with a perfectly liquid bond market, both the implied Bond-CDS spread and bid-ask spread will be zero.

Since it is well-known that CDS market is most liquid for 5-year contracts, our calibration focuses on the dimension of 5-year maturity. Thanks to endogenous liquidity built in the model, our calibrated model matches well the empirical patterns of default probabilities and credit spreads for 5-year bonds both in the cross section (across credit ratings) and over time (across business cycles).[3] On the dimension of non-default risk, endogenously linking

[3]Our calibration on aggregate macroeconomic states focuses on normal expansions and recessions, but not crises. As a result, in constructing empirical moments for recessions, we exclude the 2008 crises period from

bond liquidity to a firm's distance-to-default allows us to generate the cross-sectional and business-cycle patterns in both Bond-CDS spreads and bid-ask spreads. Overall, our model produces quantitatively reasonable non-default risk for corporate bonds. Relative to the data, our model can explain about 55% of observed variation in Bond-CDS spreads across rating classes, and future research incorporating heterogeneity of investor bases should help in this regard. Finally, the matching on 10-year bonds is less satisfactory, in that our model features a much steeper term structure of credit spreads and Bond-CDS spreads than the data suggests.

Our model provides new insights on the roles of default and liquidity in determining a firm's borrowing cost. A common practice in the empirical literature is to decompose credit spreads into a liquidity and a default component, with the interpretation that these components are independent of each other. Our model suggests that both liquidity and default are endogenously linked, and thus there can be economically important interactions. Such dynamic interactions are difficult to capture using reduced-form models with exogenously imposed liquidity premia.

Furthermore, we propose a structural decomposition that nests the common additive default-liquidity decomposition to quantify the interaction between default and liquidity for corporate bonds. Motivated by Longstaff, Mithal, and Neis (2005) who use CDS spread to proxy for default risk, we identify the "default" part by pricing a bond in a counterfactually perfectly liquid market but with the model implied default threshold. After subtracting this "default" part, we identify the remaining credit spread as the "liquidity" part. We further decompose the "default" ("liquidity") part into a "pure default" ("pure liquidity") component and a "liquidity-driven-default" ("default-driven liquidity") component, where the "pure default" or "pure liquidity" part is the spread implied by a counterfactual model where either the bond market is perfectly liquid as in Leland and Toft (1996) hence equity holders default less often, or only the over-the-counter search friction is at work for default-free bonds as in Duffie, Gârleanu, and Pedersen (2005), respectively. The two interaction terms that emerge, i.e., the "liquidity-driven default" and the "default-driven liquidity" components, capture

October 2008 to March 2009 throughout.

the endogenous positive spiral between default and liquidity. For instance, "liquidity-driven-default" is driven by the rollover risk mechanism in that firms relying on finite-maturity debt financing will default earlier when facing worsening secondary market liquidity. We show that these interaction terms are significant across all ratings, which accounts for 25%∼30% of credit spread for Aaa/Aa rated bonds, while 35%∼40% of credit spreads for Ba rated bonds.

Besides giving a more complete picture of how the default and liquidity forces affect credit spreads, the interconnection between liquidity and default also has important implications for the ongoing debate regarding how should accounting standards recognize losses on financial assets. The interesting interplay between default and liquidity and their respective accounting recognitions (e.g., being recognized as net income vs. comprehensive income) has been illustrated in the collapse of Asset-Backed-Securities market during the second half of 2007. Our model suggests that (il)liquidity can affect the credit losses for these debt instruments. Moreover, it offers a framework on how to use the liquidity information to evaluate the expected credit-losses, which will help with the effort to improve the accounting for credit losses for debt instruments.

Finally, our structural decomposition also offers important insight on evaluating hypothetical government policies, as it is important to fully take into account of how an individual firm's default responds to liquidity conditions. Imagine a policy that makes the secondary market in recession as liquid as in normal times, which lowers the credit spread of Ba rated bonds in recession by about 237 bps (about 49% of the spread). Our decomposition implies that the pure liquidity component only explains 10% of the reduction in credit spreads. The liquidity-driven default part, which captures lower default risk from firms with mitigated rollover losses, can explain 45% of this drop. The default-driven liquidity part, which captures the endogenous reduction of liquidity premium for safer bonds, can also explain about 45%. The prevailing view in the literature masks this interdependence between default and liquidity and thus tends to miss these interaction terms.

The paper is structured as follows. Section 2 introduces the model, which is solved in Section 3. Section 4 presents the main calibration. Section 5 discusses the model-based default-liquidity decomposition, and analyzes the effectiveness of a policy geared towards liquidity provision from the perspective of our decomposition. Section 6 concludes. The appendix provides proofs and a more general formulation of the model.

2. The Model

2.1 Aggregate States and the Firm

The following model elements are similar to Chen (2010), except that we study the case in which firms issue bonds with an average finite maturity a la Leland (1998) so that rollover risk in He and Xiong (2012) is present.

2.1.1 Aggregate states and stochastic discount factor

The aggregate state of the economy is described by a continuous time Markov chain, with the current Markov state denoted by s_t and the physical transition density between state i and state j denoted by $\zeta_{ij}^{\mathcal{P}}$. We assume an exogenous stochastic discount factor (SDF):

$$\frac{d\Lambda_t}{\Lambda_t} = -r(s_t)dt - \eta\left(s_t\right)dZ_t^m + \sum_{s_t \neq s_{t-}} \left(e^{\kappa\left(s_{t-},s_t\right)} - 1\right) dM_t^{\left(s_{t-},s_t\right)}, \tag{1}$$

where $\eta\left(\cdot\right)$ is the state-dependent price of risk for Brownian shocks, $dM_t^{(j,k)}$ is a compensated Poison process capturing switches between states, and $\kappa\left(i,j\right)$ embeds the jump risk premia such that in the risk neutral measure, the distorted jump intensity between states is $\zeta_{ij}^{\mathcal{Q}} = e^{\kappa(i,j)}\zeta_{ij}^{\mathcal{P}}$. In this paper we focus on the case with binary aggregate states to capture the notion of economic expansions and recessions, i.e., $s_t \in \{G, B\}$. In the Appendix we provide the general setup for the case with $n > 2$ aggregate states.

Later on, we will introduce undiversifiable idiosyncratic liquidity shocks for investors in the model. Upon receiving a liquidity shock, an investor who cannot sell the bond he is holding immediately will incur a positive holding cost. In the presence of such liquidity shocks, bond investors can still price assets using the SDF in (1) provided that the bond holding is infinitesimal in the representative investor's portfolio.[4]

2.1.2 Firm cash flows and risk neutral measure

A firm has assets in place that generate cash flows at the rate of Y_t. Under the physical measure $\mathcal{P}$, the cash-flow rate Y_t follows, given the aggregate state s_t,

$$\frac{dY_t}{Y_t} = \mu_{\mathcal{P}}\left(s_t\right) dt + \sigma_m\left(s_t\right) dZ_t^m + \sigma_f dZ_t^f. \tag{2}$$

Here, dZ_t^m captures aggregate Brownian risk, while dZ_t^f captures idiosyncratic Brownian risk. Given the stochastic discount factor Λ_t, the cash-flow dynamics under the risk neutral measure $\mathcal{Q}$ are given by

$$\frac{dY_t}{Y_t} = \mu_{\mathcal{Q}}\left(s\right) dt + \sigma\left(s\right) dZ_t^{\mathcal{Q}},$$

where $Z_t^{\mathcal{Q}}$ is a Brownian Motion under the risk-neutral measure $\mathcal{Q}$. The state-dependent risk-neutral cash-flow drift and volatility are given by

$$\mu_{\mathcal{Q}}^s \equiv \mu_{\mathcal{P}}\left(s\right) - \sigma_m\left(s\right)\eta\left(s\right), \text{ and } \sigma_s \equiv \sqrt{\sigma_m^2\left(s\right) + \sigma_f^2}.$$

For the ease of notation, we work with log cash flows $y \equiv \log\left(Y\right)$ throughout. Define

$$\mu_s \equiv \mu_{\mathcal{Q}}^s - \frac{1}{2}\sigma_s^2 = \mu_{\mathcal{P}}\left(s\right) - \sigma_m\left(s\right)\eta\left(s\right) - \frac{1}{2}\left(\sigma_m^2\left(s\right) + \sigma_f^2\right),$$

so that we have

$$dy_t = \mu_s dt + \sigma_s dZ_t^{\mathcal{Q}}. \tag{3}$$

From here on, we work under measure $\mathcal{Q}$ unless otherwise stated, so we drop the superscript $\mathcal{Q}$ in $dZ_t^{\mathcal{Q}}$ and $\zeta_{ij}^{\mathcal{Q}}$ to simply write dZ_t and ζ_{ij} where no confusion can arise.

As is standard in the asset pricing literature, we can obtain valuations for any asset as

[4] Intuitively, if the representative agent's consumption pattern is not affected by the idiosyncratic shock brought on by the bond holdings (which is true if the bond holding is infinitesimal relative to the rest of the portfolio), then the representative agent's pricing kernel is independent of idiosyncratic undiversified shocks.

the expected cash flows discounted with the risk-free rate under the risk neutral measure $\mathcal{Q}$. The unlevered firm value, given the aggregate state s and the cash-flow rate y, is

$$\mathbf{v}_U(y) \equiv \begin{bmatrix} r_G - \mu_G + \zeta_G & -\zeta_G \\ -\zeta_B & r_B - \mu_B + \zeta_B \end{bmatrix}^{-1} \mathbf{1} \exp(y) \,. \tag{4}$$

We will use v_U^s to denote the element of $\mathbf{v}_U$ in state s. Is v_U the price-dividend ratio? If so, remove y.

2.1.3 Firm's debt maturity structure and rollover frequency

The firm has bonds in place of measure 1 which are identical except for their time to maturity, and thus the aggregate and individual bond coupon (face value) is c (p). As in Leland (1998), equity holders commit to keeping the aggregate coupon and outstanding face-value constant before default, and thus issue new bonds of the same average maturity as the bonds maturing.

Each bond matures with intensity m, and the maturity event is i.i.d. across individual bonds. Thus, by law of large numbers over $[t, t + dt)$ the firm retires a fraction $m \cdot dt$ of its bonds. This implies an expected average debt maturity of $\frac{1}{m}$. The deeper implication of this assumption is that the firm adopts a "smooth" debt maturity structure with an average refinancing/rollover frequency of m. As shown later, the rollover frequency (at the firm level) is an important factor in how secondary market liquidity affects a firm's endogenous default decisions.

2.2 Secondary Over-the-Counter Corporate Bond Market

We follow Duffie, Gârleanu, and Pedersen (2005) and He and Milbradt (2014) to model the over-the-counter corporate bond market. Bond investors can hold either zero or one unit of the bond. They start in the H state without any holding cost when purchasing corporate bonds in the primary market. As time passes by, H-type bond holders are hit by idiosyncratic liquidity shocks with intensity ξ_s. These liquidity shocks lead them to become L-types who bear a positive holding cost hc_s per unit of time, which we micro-found shortly (instead of treating them as entirely exogenous, as in He and Milbradt (2014)). All trades have to be intermediated through dealers, and we derive the trading equilibrium at time t similar to Duffie, Gârleanu, and Pedersen (2005).

2.2.1 Endogenous Holding Cost

Different from He and Milbradt (2014) who treat the holding costs hc_s as some exogenous constants, we micro-found the holding cost (hc_s) to be dependent on prevailing bond prices as follows:

$$hc_s = \chi_s \left[N - P^s\left(y\right)\right] \tag{5}$$

where $N > 0, \chi_s > 0$ are constants and $P^s\left(y\right)$ is the endogenous market price of the bond (to be derived in the next section).

In Appendix B, we provide a micro-foundation for (5) based on collateralized borrowing. There, we interpret the occurrence of liquidity shock as the urgent need for an investor to raise cash (which exceeds the value of all the liquid assets he holds), a common phenomenon in modern financial institutions. Bond investors can first use their bond holdings as collateral to raise collateralized financing at the risk-free rate but subject to a haircut; any remaining gap must be financed through uncollateralized borrowing, which requires a higher financing cost. In this setting, the investor obtains less collateralized financing if the current market price of the bond is lower or the haircut for the bond is higher (which occurs exactly when the bond price goes down). The investor's effective holding cost is then given by the additional uncollateralized financing cost, which increases when the bond price goes down and will take the linear form in (5) under certain functional form of haircuts.

In Equation (5), at issuance the bond is priced at par (say $p = 100$), implying a baseline holding cost of $\chi_s\left(N - p\right)$. With $\chi_s > 0$, the holding cost increases as the firm moves closer to default, so that the bond market value $P^s\left(y\right)$ goes down. This is the key channel through which our model captures the empirical pattern that lower rated bonds are with worse secondary market liquidity.

The holding cost hc_s in (5) also depends on the aggregate state, through the following two channels. First, we will set $\chi_B > \chi_G$, which can be justified by the fact that the wedge between the collateralized and uncollateralized borrowing costs is higher in bad times. Second, the bond value $P^s(y)$ drops in bad times, giving rise to a higher holding cost.

While we provide one micro-foundation for hc_s based on collateralized borrowing, there are other mechanisms through which institutional investors hit by liquidity shock incur extra losses if the market value of their bond holdings has dropped. For instance, suppose that corporate bond fund managers face some unexpected withdrawals when the liquidity shock hit. As a model with learning about uncertain managerial skills would suggest, the deteriorating bond portfolios can trigger even greater fund outflows and extra liquidation costs.

2.2.2 Dealers and Equilibrium Prices

There is a trading friction in moving the bonds from L-type sellers to H-type buyers without bond holdings, in that trades have to be intermediated by dealers in the over-the-counter market. Sellers meet dealers with intensity λ_s, which we interpret as the intermediation intensity of the financial sector. For simplicity, we assume that after L-type investors sell their holdings, they exit the market forever. The H-type buyers on the sideline currently not holding the bond also contact dealers with intensity λ_s. We follow Duffie, Gârleanu, and Pedersen (2007) to assume Nash-bargaining weights β for the investor and $1-\beta$ for the dealer across all dealer-investor pairs.

Dealers use the competitive (and instantaneous) interdealer market to sell or buy bonds. When a contact between a type L seller and a dealer occurs, the dealer can instantaneously sell a bond at a price M to another dealer who is in contact with an H investor via the interdealer market. If he does so, the bond travels from an L investor to an H investor via the help of the two dealers who are connected in the inter-dealer market.

Fixing any aggregate state s, denote by D_l^s the individual bond valuation for the investor with type $l \in \{H, L\}$. Denote by B^s the bid price at which the L type is selling his bond, by A^s the ask price at which the H type is purchasing this bond, and by M^s the inter-dealer market price.

Following He and Milbradt (2014), we assume that the flow of H-type buyers contacting dealers is greater than the flows of L-type sellers contacting dealers; in other words, the secondary market is a *seller's market*. Similar to Duffie, Gârleanu, and Pedersen (2005) and He and Milbradt (2014), we have the following proposition. Essentially, Bertrand competition, the holding restriction, and excess demand from buyer-dealer pairs in the interdealer market drive the surplus of buyer-dealer pairs to zero.[5]

Proposition 1. *Fix valuations D_H^s and D_L^s, and denote the surplus from trade by $\Pi^s = D_H^s - D_L^s > 0$. In equilibrium, the ask price A^s and inter-dealer market price M^s are equal to D_H^s, and the bid price is given by $B^s = \beta D_H^s + (1-\beta) D_L^s$. The dollar bid ask spread is $A^s - B^s = (1-\beta)(D_H^s - D_L^s) = (1-\beta)\Pi^s$.*

As is typical in any model with decentralized trading, there is no single "market price" in our over-the-counter market. However, we use "market price" when we model the endogenous holding cost in equation (5). For simplicity, which is also consistent with market practice, we take the mid-price between the bid and the ask prices, i.e.,

$$P^s = \frac{A^s + B^s}{2} = \frac{(1+\beta) D_H^s + (1-\beta) D_L^s}{2}. \tag{6}$$

Critically, the holding cost hc_s is linear in individual bond valuations, i.e.,

$$hc_s = \chi_s \left(N - \frac{(1+\beta) D_H^s + (1-\beta) D_L^s}{2} \right), \tag{7}$$

which allows us to derive bond values D_l^s in closed-form later.

Finally, empirical studies focus on the proportional bid-ask spread which is defined as the dollar bid-ask spread divided by the mid price, i.e.,

$$\Delta^s(y,\tau) = \frac{2(1-\beta)(D_H^s - D_L^s)}{(1+\beta) D_H^s + (1-\beta) D_L^s} = \frac{(1-\beta)\Pi^s}{D_H^s - \frac{1-\beta}{2}\Pi^s}. \tag{8}$$

[5] This further implies that the value function of buyers without bond holdings who are sitting on the sideline is identically zero, which makes the model tractable. Introducing for example direct bilateral trades or assuming a *buyer's market* would entail tracking the value functions of investors on the sideline but would not add additional economic insights pertaining to credit risk in particular.

2.3 Delayed Bankruptcy Payouts and Effective Recovery Rates

When the firm's cash flow deteriorates, equity holders are willing to repay the maturing debt holders only when the equity value is still positive, i.e. the option value of keeping the firm alive justifies absorbing rollover losses and coupon payments. The firm defaults when its equity value drops to zero at some default threshold y_{def}, which is endogenously chosen by equity holders. The bankruptcy costs is a fraction $1-\hat{\alpha}_s$ of the value from unlevered assets $v_U^s(y_{def})$ given in (4), and the debt holder's bankruptcy recovery $\hat{\alpha}_s$ may depend on the aggregate state s.

As emphasized in He and Milbradt (2014), if bankruptcy leads investors to receive the bankruptcy proceeds immediately, then bankruptcy confers a "liquidity" benefit similar to a bond maturing. This "expedited payment" benefit runs counter to the fact that in practice bankruptcy leads to the freezing of assets within the company and a delay in the payout of any cash depending on court proceeding.[6] Moreover, bond investors with defaulted bonds may face a much more illiquid secondary market (e.g., Jankowitsch, Nagler, and Subrahmanyam (2013)), and potentially a much higher holding cost once liquidity shocks hit due to regulatory or charter restrictions which prohibit institutions to hold defaulted bonds.

To capture above features, following He and Milbradt (2014) we assume that a bankruptcy court delay leads the bankruptcy cash payout $\hat{\alpha}_s v_U^s < p$ to occur at a Poisson arrival time with intensity θ,[7] where we simply denote $v_U^s(y_{def})$ by v_U^s. The holding cost of defaulted bonds for L-type investors is $hc_{def}^s v_U^s$ where $hc_{def}^s > 0$, and the secondary over-the-counter market for defaulted bonds is illiquid with intermediation intensity λ_{def}^s. The type- and state-dependent bond recovery at the time of default can be written as: (see Appendix)

$$\mathbf{D}^{def}(y) = \left[\alpha_H^G v_U^G(y), \alpha_L^G v_U^G(y), \alpha_H^B v_U^B(y), \alpha_L^G v_U^B(y)\right]^T \tag{9}$$

[6]For evidence on inefficient delay of bankruptcy resolution, see Gilson, John, and Lang (1990) and Ivashina, Smith, and Iverson (2013). In addition, the Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after about three and a half years.

[7]We could allow for a state-dependent bankruptcy court delay, i.e., $\theta(s)$; but the Moody's Ultimate Recovery Dataset reveals that there is little difference between the recovery time in good time versus bad time.

For easier comparison to existing Leland-type models where debt recovery at bankruptcy is simply $\hat{\alpha} v_U$, we denote $\boldsymbol{\alpha} \equiv \left[\alpha_H^G, \alpha_L^G, \alpha_H^B, \alpha_L^B\right]^\top$ as the *effective* bankruptcy recovery rates at the time of default. These effective bankruptcy recovery factors $\boldsymbol{\alpha}$ are determined by the post-default corporate bond market structures; and they are the only critical ingredients for us to solve for the pre-default bond valuations liquidity. In calibration, we will not rely on deeper structural parameters (say, post-default holding cost hc_{def}). Instead, we choose these effective recovery rates $\boldsymbol{\alpha}$ to target both the market price of defaulted bonds observed immediately after default (which are close to L-type valuations) and the associated empirical bid-ask spreads.

3. Model Solutions

Denote by $D_l^{(s)}$ the l-type bond value in aggregate state s, $E_l^{(s)}$ the equity value in aggregate state s, and $\boldsymbol{y}_{def} = \left[y_{def}^G, y_{def}^B\right]^\top$ the vector of endogenous default boundaries. We derive the closed-form solution for debt and equity valuations in this section as a function of a given $\boldsymbol{y}_{def}$, along with the characterization of endogenous default boundaries $\boldsymbol{y}_{def}$.

3.1 Debt Valuations

Because equity holders will default earlier in state B, i.e., $y_{def}^G < y_{def}^B$, the domains of debt valuations change when the aggregate state switches. We deal with this issue by the following treatment; see the Appendix for the generalization of this analysis.

Define two intervals $I_1 = \left[y_{def}^G, y_{def}^B\right]$ and $I_2 = \left[y_{def}^B, \infty\right)$, and denote by $D_l^{s,i}$ the restriction of D_l^s to the interval I_i, i.e., $D_l^{s,i}(y) = D_l^s(y)$ for $y \in I_i$. Clearly, $D_l^{B,1}(y) = \alpha_l^B v_U^B(y)$ is in the "dead" state, so that the firm immediately defaults in interval I_1 when switching into state B (from state G). In light of this observation, on interval $I_2 = \left[y_{def}^B, \infty\right)$ all bond valuations denoted by $\mathbf{D}^{(2)} = \left[D_H^{G,2}, D_L^{G,2}, D_H^{B,2}, D_L^{B,2}\right]^\top$ are "alive."

For bond valuations, we simply treat holding costs given liquidity shocks as negative dividends, which effectively lower the coupon flows that investors are receiving. Moreover, we directly shocks, which is justified by the assumption that the illiquid bond holding is infinitesimal in the representative investor's portfolio. For further discussions, see footnote 4 and the end of Section 2.1.2.

The bond valuation equation can be written as following matrix form:

$$\begin{aligned}\underbrace{\hat{\mathbf{R}}\cdot\mathbf{D}^{(2)}(y)}_{Discounting,4\times1} &= \underbrace{\boldsymbol{\mu}}_{4\times4}\underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times1}+\frac{1}{2}\underbrace{\boldsymbol{\Sigma}}_{4\times4}\underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times1}+\underbrace{\hat{\mathbf{Q}}\cdot\mathbf{D}^{(2)}(y)}_{Transition,4\times1}\\ &+\underbrace{c\mathbf{1}_4}_{Coupon,4\times1}+\underbrace{m\left[p\mathbf{1}_4-\mathbf{D}^{(2)}(y)\right]}_{Maturity,4\times1}-\underbrace{\left[\boldsymbol{\chi}\cdot\mathbf{N}-\boldsymbol{\chi}\cdot\mathbf{W}\cdot\mathbf{D}^{(2)}(y)\right]}_{Holding\,Cost,4\times1},\end{aligned}\tag{10}$$

where $\hat{\mathbf{R}},\boldsymbol{\mu},\boldsymbol{\Sigma},\hat{\mathbf{Q}},\boldsymbol{\chi}$, and $\mathbf{W}$ are given in the Appendix. Here, the left-hand-side is the required return of holding the bond. On the right-hand-side, the first two terms capture the evolution of cash flow state, while the third term gives the switching of Markov aggregate states. The fourth term is the coupon payment, and the fifth term captures debt maturing. The last term gives the holding cost. Importantly, as in (7), endogenous debt valuations enter the holding cost linearly. The following proposition gives the closed-form solution for debt valuations.

Proposition 2. *The bond values on interval i are given by*

$$\underbrace{\mathbf{D}^{(i)}}_{2i\times1}=\underbrace{\mathbf{G}^{(i)}}_{2i\times4i}\cdot\underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times4i}\cdot\underbrace{\mathbf{b}^{(i)}}_{4i\times1}+\underbrace{\mathbf{k}_0^{(i)}}_{2i\times1}+\underbrace{\mathbf{k}_1^{(i)}}_{2i\times1}\exp(y),\tag{11}$$

where the matrices $\mathbf{G}^{(i)}$, $\boldsymbol{\Gamma}^{(i)}$ and the vectors $\mathbf{k}_0^{(i)}$, $\mathbf{k}_1^{(i)}$ and $\mathbf{b}^{(i)}$ are given in the Appendix A.

3.2 Equity Valuations and Default Boundaries

When the firm refinances its maturing bonds, we assume that the firm can place newly issued bonds with H investors in a competitive primary market.[8] This implies that there are rollover gains/losses of $m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y)-p\mathbf{1}_i\right]$ at each instant as a mass $m\cdot dt$ of debt holders matures on $[t,t+dt]$, where $\mathbf{S}^{(i)}$ is a $i\times2i$ matrix that selects the appropriate D_H as we assumed the firm

[8] This is consistent with our seller's market assumption in Section 2.2, i.e., there are sufficient H-type buyers waiting on the sidelines.

issues to H-type investors in the primary market. For instance, for $y \in I_2 = [y_{def}(B), \infty)$, we have $\mathbf{D}^{(2)} = \left[D_H^{G,2}, D_L^{G,2}, D_H^{B,2}, D_L^{B,2}\right]^\top$ and $\mathbf{S}^{(2)} = (1-\omega)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, where $\omega \in (0,1)$ is the proportional issuance costs in the primary corporate bond market.

The rollover term due to bond repricing enters the equity valuation. For ease of exposition, we denote by double letters (e.g. $\mathbf{xx}$) a constant for equity that takes a similar place as a single letter (i.e. $\mathbf{x}$) constant for debt. We can write down the valuation equation for equity on interval I_i. For instance, on interval I_2 we have

$$\begin{aligned} \underbrace{\mathbf{RR}\cdot\mathbf{E}^{(2)}(y)}_{Discounting,2\times1} &= \underbrace{\boldsymbol{\mu\mu}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)'(y)}_{2\times1}+\frac{1}{2}\underbrace{\boldsymbol{\Sigma\Sigma}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)''(y)}_{2\times1}+\underbrace{\mathbf{QQ}\cdot\mathbf{E}^{(2)}(y)}_{Transition,2\times1} \\ &\quad + \underbrace{\mathbf{1}_2\exp(y)}_{Cashflow,2\times1} - \underbrace{(1-\pi)\,c\mathbf{1}_2}_{Coupon,2\times1} + \underbrace{m\left[\mathbf{S}^{(2)}\cdot\mathbf{D}^{(2)}(y)-p\mathbf{1}_2\right]}_{Rollover,2\times1} \end{aligned} \tag{12}$$

where the matrices $\boldsymbol{\mu\mu}$, $\boldsymbol{\Sigma\Sigma}$, $\mathbf{QQ}$, and $\mathbf{RR}$ are given in the Appendix.

Proposition 3. *The equity value is given by*

$$\underbrace{\mathbf{E}^{(i)}(y)}_{i\times1} = \underbrace{\mathbf{GG}^{(i)}}_{i\times2i}\cdot\underbrace{\exp\left(\boldsymbol{\Gamma\Gamma}^{(i)}y\right)}_{2i\times2i}\cdot\underbrace{\mathbf{bb}^{(i)}}_{2i\times1}+\underbrace{\mathbf{KK}^{(i)}}_{i\times4i}\underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times4i}\underbrace{\mathbf{b}^{(i)}}_{4i\times i}+\underbrace{\mathbf{kk}_0^{(i)}}_{i\times1}+\underbrace{\mathbf{kk}_1^{(i)}}_{i\times1}\exp(y) \quad \text{for } y \in I_i \tag{13}$$

where the matrices $\mathbf{GG}^{(i)}$, $\boldsymbol{\Gamma\Gamma}^{(i)}$, $\mathbf{KK}^{(i)}$, $\boldsymbol{\Gamma}^{(i)}$ *and the vectors* $\mathbf{kk}_0^{(i)}$, $\mathbf{kk}_1^{(i)}$ *and* $\mathbf{b}^{(i)}$ *are given in the Appendix A.*

Finally, the endogenous bankruptcy boundaries $\boldsymbol{y}_{def} = \left[y_{def}^G, y_{def}^B\right]^\top$ are given by the standard smooth-pasting condition:

$$\left(\mathbf{E}^{(1)}\right)'\left(y_{def}^G\right)_{[1]} = 0, \text{ and } \left(\mathbf{E}^{(2)}\right)'\left(y_{def}^B\right)_{[2]} = 0. \tag{14}$$

3.3 Model Implied Credit Default Swap

One of key empirical moments for bond liquidity used in this paper is the Bond-CDS spread, defined as Bond credit spread minus the spread of the corresponding Credit Default Swap (CDS). Since the CDS market is much more liquid than that of corporate bonds, following

Longstaff, Mithal, and Neis (2005) we compute the model implied CDS spread under the assumption that the CDS market is perfectly liquid.[9]

Let τ (in years from today) be the time of default. Formally, $\tau \equiv \inf\{t : y_t \leq y_{def}^{s_t}\}$ can be either the first time at which the cash-flow rate y_t reaches the default boundary y_{def}^{s} in state s, or when $y_{def}^{G} < y_t < y_{def}^{B}$ so that a change of state from G to B triggers default. Thus, for a T-year CDS contract, the required flow payment f is the solution to the following equation:

$$\mathbb{E}^{\mathcal{Q}}\left[\int_0^{\min[\tau,T]} \exp\left(-rt\right) f dt\right] = \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-r\tau 1_{\{\tau\leq T\}}\right) LGD_\tau\right], \tag{15}$$

where LGD_τ is the loss-given-default when the default occurs at time τ. If there is no default, no loss-given-default is paid out by the CDS seller. The loss-given-default LGD is defined as the bond face value p minus its recovery value, and we follow the practice to define the recovery value as the transaction price right after default (with the mid price when the firm defaults at y_{def}^{s}). We calculate the required flow payment f that solves (15) using a simulation method. Finally, the CDS spread, f/p, is defined as the ratio between the flow payment f and the bond's face value p.

4. Calibration

4.1 Benchmark Parameters

We calibrate the parameters governing firm fundamentals and pricing kernels to the key moments of the aggregate economy and asset pricing. Parameters governing time-varying liquidity conditions are calibrated to their empirical counterparts on bond turnover, dealer's bargaining power, and observed bid-ask spreads.

[TABLE 1 ABOUT HERE]

[9]Arguably, the presence of CDS market will in general affect the liquidity of corporate bond market; but we do not consider this effect. A recent theoretical investigation by Oehmke and Zawadowski (2013) shows ambiguous results on this regard.

4.1.1 SDF and cash flows liquidity parameters

We follow Chen, Xu, and Yang (2012) in calibrating firm fundamentals and investors' pricing kernel. Table 1 reports the benchmark parameters we use, which are standard in the literature. Start from investors' pricing kernel. The risk free rate is $r_G = r_B = 2\%$ in both aggregate states, so that we abstract from the standard term structure effect. For simplicity, in this paper we do not model the liquidity premium for Treasuries and compute the credit spreads relative to the risk-free rate instead of the Treasury yields.[10] Transition intensities give the duration of the business cycle (10 years for expansions and 2 years for recessions). Jump risk premium $\exp(\kappa) = 2$ in state G (and the state B jump risk premium is the reciprocal of that of state G) is consistent with a long-run risk model with Markov-switching conditional moments and calibrated to match the equity premium (Chen (2010)). The risk price η is the product of relative risk aversion γ and consumption volatility σ_c: $\eta = 0.165$ (0.255) in state G (state B) requires $\gamma = 10$ and $\sigma_c = 1.65\%$ ($\sigma_c = 2.55\%$).

On the firm side, the cash-flow growth is matched to the average growth rate of aggregate corporate profits. State-dependent systematic volatilities σ_m^s are chosen to match equity return volatilities. We set $m = 0.2$ so that the average debt maturity is about $1/m = 5$ years. This is close to the empirical median debt maturity (including bank loans and public bonds) reported in Chen, Xu, and Yang (2012). We set the debt issuance cost ω in the primary corporate bond market to be 1% as in Chen (2010). And, the idiosyncratic volatility σ_i is chosen to match the default probability of Baa firms. There is no state-dependence of σ_i as we do not have data counterparts for state-dependent Baa default probabilities. Finally, as explained later, the firm's cash-flow is determined from empirical leverage observed in the data.

Chen, Collin-Dufresne, and Goldstein (2009) argue that generating a reasonable equity Sharpe ratio is an important criterion for a model that tries to simultaneously match the

[10] It has been widely recognized (e.g., Duffie (1996), Krishnamurthy (2002), Longstaff (2004)) that Treasury yields are lower than the risk-free rate due to their special role in financial markets. In unreported results, we introduce (exogenous) state-dependent liquidity premia Δ_s for Treasuries and calibrate them using the spreads between 3-month general collateral repo rates and Treasury yields. This introduction of the Δ_s simply leads to level shifts between the spreads using the risk-free rate as the benchmark and those using the Treasury yield as benchmark.

default rates and credit spreads, for otherwise one can simply raise credit spreads by imposing unrealistically high systematic volatility and prices of risk. Based on our calibration (especially the choices of σ_m, σ_i, κ, and η), we obtain the equity Sharpe ratio of 0.11 in state G and 0.20 in state B, which is close to the mean firm-level Sharpe ratio for the whole universe of the CRSP firms (0.17) reported in Chen, Collin-Dufresne, and Goldstein (2009).

4.1.2 Bond market illiquidity

The liquidity parameters in secondary corporate bond market are less standard in the literature. We first fix the state-dependent intermediary meeting intensity based on anecdotal evidence, so that it takes a bond holder on average a week ($\lambda_G = 50$) in the good state and 2.6 weeks ($\lambda_B = 20$) in the bad state to find an intermediary to sell all bond holdings.[11] We interpret the lower λ in state B as a weakening of the financial system and its ability to intermediate trades. We then set bond holders bargaining power $\beta = 0.05$ independent of the aggregate state, based on the empirical work that estimates search frictions in secondary corporate bond markets (Feldhütter (2012)).

We choose intensity of liquidity shocks, ξ_s, based on observed bond turnovers in the secondary market. In the TRACE sample from 2005 to 2010, the value-weighted turnover of corporate bonds during NBER recessions is about 0.7 times per year, and there are no significant difference for bond turnovers over business cycle. However, on the liquidity dimension, we are aiming to explain the observed Bond-CDS spreads. There are significantly less firms with CDS contracts traded (a total of 1,313 firms), and their bond turnover rate is much higher than the whole sample. Their value-weighted turnover from 2005 to 2010 is about 3.1 per year; over business cycle, the normal time turnover is 3.3 per year while the recession time turnover is around 2 times per year.

In our model, given our choice of relative large meeting intensities (λ_s), the turnover rate is almost determined by the liquidity shock intensity ξ_s.[12] More importantly, all turnovers in secondary corporate bond markets are driven by liquidity reasons. Apparently, in

[11] Ideally one can infer λ using the total time the corporate bond funds take to complete a sale, which is a challenging task empirically.

[12] The expected turnover is $\frac{\xi_s \lambda_s}{\xi_s + \lambda_s} \simeq \xi_s$ when $\lambda_s \gg \xi_s$.

practice investors trade corporate bonds for reasons other than liquidity, and during recession institutional investors are more likely to be hit by liquidity shocks and hence trade their bond holdings. We thus rely on the empirical turnover frequency during recessions to set $\xi_B = 2$. For simplicity, we then fix the the state-G liquidity intensity $\xi_G = 2$ as well to reduce the number of free parameters.

The parameters χ_s's in equation (5) are central in determining the endogenous holding costs and thus illiquidity of corporate bonds in secondary market. We calibrate $\chi_G = 0.12$ and $\chi_G = 0.17$ to target the bid-ask spread for bonds with investment grade in both aggregate states. In light of the particular micro-foundation in Section 2.2.1 where χ_s is interpreted as the wedge between collateralized and uncollateralized borrowing costs, both numbers are higher than typical observed TED (LIBOR-Tbill) spreads. However, TED spreads (using uncollateralized rate among a select of large and reputable banks) might be an underestimate of the true cost of uncollateralized borrowing. More importantly, as a quantitative paper, the state-dependent holding cost parameters χ_s in (5) are in reduced form and meant to capture more than this particular micro-foundation of uncollateralized borrowing (see discussion in Section 2.2.1). Finally, we set $N = 107$ (with par bond value of $p = 100$) to roughly target the Bond-CDS spread for Baa rated bonds in state G. [13]

4.1.3 Effective recovery rates

As explained in Section 2.3, our model features type- and state-dependent recovery rates α_l^s for $l \in \{L, H\}$ and $s \in \{G, B\}$. We first borrow from the existing structural credit risk literature (say, Chen (2010)) who treats the traded prices right after default as recovery rates, and estimates recovery rates of $57.6\% \cdot v_U^G$ in normal times and $30.6\% \cdot v_u^B$ in recessions (recall v_U^s is the unlevered firm value at state s).

Assuming that post-default prices are bid prices at which investors are selling, then

[13] In the micro-foundation provided in Appendix XX, given other baseline calibrations, the choice of $N = 107$ and χ_s gives rise to model-implied haircuts that range from 10% to 25% depending on ratings. They are broadly consistent with BIS (2010) which gives a comprehensive survey conducted by the BIS Committee on the Global Financial System (including both Europe and US) for June 2007 and June 2009. The model implied haircuts are 9% for Aaa/Aa bonds, 10% for A, 12% for Baa, and 18% for Ba. In BIS (2010), the average haircuts for non-rated counterparties are 6.7% for Aaa/Aa rated bonds, 12% for Baa, and 23% for high yield bonds.

Proposition 1 implies:

$$0.5755 \;=\; \alpha_L^G + \beta(\alpha_H^G - \alpha_L^G), \text{ and } 0.3060 = \alpha_L^B + \beta(\alpha_H^B - \alpha_L^B). \tag{16}$$

We need two more pieces of bid-ask information for defaulted bonds to pin down the α_l^s's. Edwards, Harris, and Piwowar (2007) report that in normal times, the transaction cost for defaulted bonds for median-sized trades is about $200bps$. To gauge the bid-ask spread for defaulted bonds during recessions, we take the following approach. Using TRACE, we first follow Bao, Pan, and Wang (2011) to calculate the implied bid-ask spreads for low rated bonds (C and below) for both non-recession and recession periods. We find that relative to the non-recession period, during recessions the implied bid-ask spread is about 3.1 times higher. Given a bid-ask spread of $200bps$ for defaulted bonds, this multiplier implies that the bid-ask spread for defaulted bonds during recessions is about $620bps$. Hence we have

$$2\% \;=\; \frac{2\,(1-\beta)\left(\alpha_H^G - \alpha_L^G\right)}{\alpha_L^G + \beta(\alpha_H^G - \alpha_L^G) + \alpha_H^G}, \text{ and } 6.2\% = \frac{2\,(1-\beta)\left(\alpha_H^B - \alpha_L^B\right)}{\alpha_L^B + \beta(\alpha_H^B - \alpha_L^B) + \alpha_H^B}. \tag{17}$$

Solving (16) and (17) gives us the estimates of:[14]

$$\boldsymbol{\alpha} = \left[\alpha_H^G = 0.5871, \alpha_L^G = 0.5749, \alpha_H^B = 0.3256, \alpha_L^B = 0.3050\right]. \tag{18}$$

4.1.4 Degree of freedom in calibration

We summarize our calibration parameters in Table 1. Although there are a total of 29 parameters, most of them are in Panel A "pre-fixed parameters" of Table 1, which are set either using the existing literature or based on moments other than the corporate bond pricing moments. We only pick (calibrate) three parameters in Panel B "calibrated parameters" freely to target the empirical moments that our model aims to explain: the idiosyncratic volatility σ_i is picked to target Baa firm default probabilities; for holding cost parameters, N is set to target the Baa state G Bond-CDS spread while χ_G and χ_B are picked to target

[14] This calculation assumes that bond transactions at default occur at the bid price. If we assume that transactions occur at the mid price, these estimates are $\alpha_H^G = 0.5813, \alpha_L^G = 0.5691, \alpha_H^B = 0.3140, \alpha_L^B = 0.2972$.

investment grade bid-ask spread in both states. As shown shortly, this degree of freedom (4) is far below the number of our empirical moments that we aim to match.

We point out that in our model, the quantitative performance along the dimension of business cycles is less surprising, simply because our model takes (and sometimes, chooses) exogenous parameters in two aggregate macroeconomic states. Because our model links the secondary bond market liquidity to the firm's distance-to-default, our model's quantitative strength is more reflected on its cross-sectional performance (say, matching the total credit spreads over four ratings). And, since we choose liquidity parameters (e.g. holding costs) to target turnover and bid-ask spreads, the matching of Bond-CDS spreads can also be considered as a success of our model.

4.2 Empirical Moments

We consider four rating classes: Aaa/Aa , A, Baa, and Ba; the first three rating classes are investment grade, while Ba is speculative grade. We combine Aaa and Aa together because there are few observations for Aaa firms. We emphasize that previous calibration studies on corporate bonds focus on the difference between Baa and Aaa only, while we are aiming to explain the level of credit spreads across a wide range of rating classes. Furthermore, we report the model performance conditional on macroeconomic states, while typical existing literature only focus on unconditional model performance (Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010), and Chen (2010)). We classify each quarter as either in "state G" or "state B" based on NBER recession. As the "B" state in our model only aims to capture normal recessions in business cycles, we exclude two quarters during the 2008 financial crisis, which are 2008Q3 and 2009Q1, to mitigate the effect caused by the unprecedented disruption in financial markets during crisis.[15]

[15] For recent empirical research that focuses on the behaviors of corporate bonds market during the 2007/08 crisis, see Dick-Nielsen, Feldhütter, and Lando (2012) and Friewald, Jankowitsch, and Subrahmanyam (2012b).

4.2.1 Default Probabilities

The default probabilities for 5-year and 10-year bonds in the data column of Panel A in Table 2 are taken from Exhibit 33 of Moody's annual report on corporate default and recovery rates (2012), which gives the cumulative default probabilities over the period of 1920-2011. Unfortunately, the state-dependent measurement on default probabilities over business cycles are unavailable.

[TABLE 2 ABOUT HERE]

4.2.2 Bond Spreads

Our data of bond spreads is obtained using Mergent Fixed Income Securities Database (FISD) trading prices from January 1994 to December 2004, and TRACE data from January 2005 to June 2012. We follow the standard data cleaning process, e.g. excluding utility and financial firms.[16] For each transaction, we calculate the bond credit spread by taking the difference between the bond yield and the treasury yield with corresponding maturity. Within each rating class, we average these observations in each month to form a monthly time series of credit spreads for that rating. We then calculate the time-series average for each rating conditional on the macroeconomic state (whether the month is classified as NBER recession), and provide the conditional standard deviation for the conditional mean. To account for the autocorrelation of these monthly series, we calculate the standard deviation using Newey-West procedure with 15 lags.

We report the conditional means for each rating and their corresponding conditional standard deviations for both 5-year and 10-year bonds in the data column in Panel B of Table 2. In the existing literature, Huang and Huang (2012) cover the period from the 1970's to the 1990's, and report an (unconditional) average credit spread of 55 bps for 4-year Aaa rated bonds, 65 bps for Aa, 96 for A, 158 for Baa, and 320 for Ba. Our unconditional 5-year average credit spreads are fairly close, which is the weighted average across conditional means

[16]For FISD data, we follow Collin-Dufresne, Goldstein, and Martin (2001). For TRACE data, we follow Dick-Nielsen (2009).

reported in Panel B of Table 2: 65 bps for Aaa/Aa, 100 bps for A, 167 for Baa, and 349 for Ba.

4.2.3 Bond-CDS spreads

Longstaff, Mithal, and Neis (2005) argue that because the market for CDS contracts is much more liquid than the secondary market for corporate bonds, the CDS spread should mainly reflect the default risk of a bond, while the credit spread also includes liquidity premium to compensate for the illiquidity in the corporate bond market. Following Longstaff, Mithal, and Neis (2005), we take the difference between the bond credit spread and the corresponding CDS spread. This Bond-CDS spread is our first empirical measure for the non-default risk of corporate bonds.

We construct Bond-CDS spreads as follows. We first match FISD bond transaction data with CDS prices from Markit, and then follow the same procedure as above, with two caveats. First, the data sample period only starts from 2005 when CDS data become available. Second, to address the potential selection issue, we follow Chen, Xu, and Yang (2012) and focus on firms that have both 5-year and 10-year bonds outstanding. The results are reported in the data column in Panel A in Table 2.

Bond-CDS spread versus Bond-CDS basis One issue is worth further discussing. Our *Bond-CDS spread* is defined as the corporate bond yield minus the treasury yield with matching maturity, and then minus its corresponding CDS spread. Another closely related measure, *Bond-CDS basis*, is of great interest to both practitioners and academic researchers. The only difference is on the risk-free benchmark: our Bond-CDS spread takes the Treasury yield as the benchmark, while Bond-CDS bases takes interest rate swap rate as the benchmark. For recent studies on Bond-CDS basis, see Gârleanu and Pedersen (2011) and Bai and Collin-Dufresne (2012).

The study of Bond-CDS basis mostly focuses on limits-to-arbitrage during the turmoil of financial market. Because interest rate swap gives a more accurate measure of an arbitrager's financing cost, the choice of interest rate swap is more appropriate when studying Bond-CDS basis.

In contrast, our paper aims to explain the credit spread, and we follow the corporate bond pricing literature in setting the Treasury yield as our benchmark. The credit spread includes both the default and liquidity components. Treasuries are a better default-free benchmark, because the interest rate swap rate is the fixed leg of LIBOR, which is contaminated by default risk. Treasuries also serve as the illiquidity-free benchmark, where "liquidity" can be interpreted broadly to include trading liquidity and market liquidity that are captured by our model.

4.2.4 Bid Ask Spreads

The second non-default measure that we study is bid-ask spreads in the secondary market for corporate bonds, whose model counterpart is given in (8). Previous empirical studies have uncovered rich patterns of bid-ask spreads across aggregate states and rating classes. More specifically, we combine Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) to construct the data counterparts for the bid-ask spread, as Edwards, Harris, and Piwowar (2007) only report the average bid-ask spread across ratings in normal times (2003-2005). The ratings considered in Edwards, Harris, and Piwowar (2007) are superior grade (Aaa/Aa) with an bid-ask spread of 40 bps, investment grade (A/Baa) with an bid-ask spread of 50 bps, and junk grade (below Ba) with a bid-ask spread of 70 bps.[17] For each grade, we then compute the measure of liquidity in Roll (1984) as in Bao, Pan, and Wang (2011), which we use to back out the bid-ask spread ratio between B-state and G-state. We multiply this ratio by the bid-ask spread estimated by Edwards, Harris, and Piwowar (2007) in normal times (2003-2005) to arrive at bid-ask spread in B state. These empirical estimates are reported in Panel B in Table 3.

[17]We take the median size trade around 240K. Edwards, Harris, and Piwowar (2007) show that trade size is an important determinants for transaction costs of corporate bonds. But, for tractability reasons, we have abstracted away from the trade size.

4.3 Model Performance on Default Risk and Credit Spreads

4.3.1 Calibration method

For any given cash-flow y, which links one-to-one to the firm's market leverage, we can compute the default probability and credit spread of bonds at 5 and 10 year maturity using Monte-Carlo methods. [18] As typical in structural corporate bond pricing models, we find that the model implied default probability and total credit spread are highly nonlinear in market leverage (see Figure 2). The non-linearity inherent in the model implies that the average credit spreads are higher than the spreads at average market leverage. We thus follow David (2008) in computing model implied aggregate moments. Specifically, we compute the market leverage (i.e., book debt over the sum of market equity and book debt) of all Compustat firms (excluding financial and utility firms and other standard filters) for which we have ratings data between 1994 and 2012.[19] We then match each firm-quarter observed in Compustat to its model counterpart based on the observed market leverage, compute the average across aggregate states, and repeat the procedure for each rating class and each maturity (5 or 10 years). Hence our model always exactly matches the data counterpart on the dimension of leverage.

Relative to the existing literature, our calibration aims at explaining the level of credit spread across ratings, rather than differences between ratings. For instance, (Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010), and Chen (2010)) focus on explaining the difference between Baa and Aaa rated bonds, which is considered as the default component of Baa rated bonds under the assumption that the observed spreads for Aaa rated bonds are mostly driven by liquidity premium. Because our framework endogenously models bond liquidity, we are able to match the credit spreads that we observe

[18] Recall that for tractability we assume that bonds are with random maturity. In calibration, we study bonds with deterministic maturities, which can be viewed as some infinitesimal bonds in the firm's aggregate debt structure analyzed in Section 2.1.3. Since the debt valuation derived in Proposition 2 does not apply, we rely on Monte-Carlo methods. Specifically, we simulate the cash flow of the firm and aggregate state for 50,000 times for a fixed duration of 5 or 10 years and count the times where the cash flow cross the state dependent default boundary and also record the cash flow received by bond holders of either H or L type. Following the literature, we consider bonds that are issued at par.

[19] A similar point is made in Bhamra, Kuehn, and Strebulaev (2010). For empirical distribution of market leverage for each rating, see Figure 1. Market leverage is defined as the ratio of book debt over book debt plus market equity.

in the data across the superior ratings (Aaa/Aa) and the high end of speculative rating bonds (say Ba).

Another important dimension that our paper improves over the existing literature is on the matching of conditional means of credit spreads. Because the success of Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010), and Chen (2010) hinges on the idea that the bond's payoff is lower in recessions with a higher marginal utility of consumption, checking whether the model implied bond spreads during recessions matches empirical counterpart can be viewed as a disciplinary test for the mechanism proposed by those papers.

4.3.2 Calibration results

Table 2 presents our calibration results on aggregate default probability (Panel A) and credit spread for bonds of four rating classes (Panel B), for both 5-year and 10-year bonds.

5-year default probabilities and credit spreads On the maturity end of 5-year, our quantitative model is able to deliver a decent matching for both cross-sectional and state-dependent patterns in default probabilities and credit spreads.

On the superior grade bonds with Aaa/Aa ratings, our model gives a reasonable matching for 5-year credit spreads: in state G, the model predicts 72.9 bps while the data counterpart is 55.7 bps; in state B, we have 99 bps in the model versus 107 bps in the data. Thanks to introducing liquidity into the structural corporate bond pricing model, we are able to produce reasonable credit spreads for Aaa/Aa bonds conditional on empirically observed default probabilities.

For lower rated bonds, the match on default probability is quite satisfactory. For instance, for Ba rated bonds, the model implied default probability is 9.9%, matching quite well with 9.8% reported by Moody's. Overall, relative to data counterpart the model implied credit spread tends to overshoot in state G while undershoot in state B, but the match are reasonable.

Our calibration exercise puts more emphasis on the 5-year horizon.[20] The reason that we focus more on 5-year, rather than 10-year, is that this paper aims to explain the non-default component of corporate bonds. The Bond-CDS spreads require the input of observed CDS spreads. Motivated by the fact that CDS market is more liquid than corporate bond market (e.g., Longstaff, Mithal, and Neis (2005)), our model assumes a perfectly liquid CDS market. In practice, it is well-known that the most liquid CDS contracts are those with a 5-year maturity. Hence, focusing on the 5-year end mitigates the potential liquidity effect of the CDS market in biasing our calibration results.

10-year default probabilities and credit spreads While our model is able to quantitatively match the cross-sectional and state-dependent pattern for the credit spreads of 5-year bonds, the matching for 10-year bonds is less satisfactory. The general pattern is that although the default probability matches the data counterpart reasonably well, the model implied conditional credit spread for 10-year bonds overshoots the empirical moments at a significant margin, suggesting that the model implied term structure of credit spreads is steeper than the data suggests. In unreported results, we find that the method of David (2008) which addresses the nonlinearity in the data (caused by the diverse distribution in leverage) has helped our model greatly to deliver a flatter term structure. This finding is consistent with Bhamra, Kuehn, and Strebulaev (2010). Nevertheless, this treatment is not strong enough to get the term structure right. Certain interesting extensions of our model (e.g., introducing jumps in cash flows that are more likely to occur in state B) should help in this dimension, and we leave future research to address this issue.

Bond recovery rates As emphasized by Huang and Huang (2012), in order for a model to explain the corporate bond spreads, it should not only be able to match the observed spreads, but also generate default probabilities and bond recovery rates that are consistent with the data. In our model, the implied bond recovery rate is 49.7% in state G and 24.5% in state B. The unconditional average recovery rate is 44.6%. These values are consistent with the

[20] More specifically, within the reasonable range used in the literature, we have chosen the state-dependent risk price η and systematic volatility σ_m to deliver an overall good match for 5-year Baa rated bonds.

average issuer-weighted bond recovery rate of 42% in Moody's recovery data over 1982-2012, and they capture the cyclical variations in recovery rates as documented in Chen (2010).

4.4 Model Performance on Non-Default Risk

Our model features an illiquid secondary market for corporate bonds, which implies that the equilibrium credit spread must compensate the bond investors for bearing not only default risk but also liquidity risk. This new element allows us to investigate the model's quantitative performance on dimensions specific to bond market liquidity, i.e., Bond-CDS spreads and bid-ask spreads, in addition to cumulative default probabilities and credit spreads on which the previous literature has focused.

4.4.1 Bond-CDS Spread

Recall that we assume a perfectly liquid CDS market in Section 3.3. In practice, although much more liquid than the secondary corporate bond market, the CDS market is still not perfectly liquid. As explained above, to mitigate this effect we have focused on bonds with 5-year maturity, because 5-year CDS contracts are traded with the most liquidity.[21]

5-year Bond-CDS spread Similar to the above procedure in Section 4.3, following David (2008) we obtain our model-implied aggregated moments by first calculate the Bond-CDS spread for each firm-quarter observation for the Bond-CDS firm sample (after 2005) based on its market leverage, conditional on the macroeconomic state.

[TABLE 3 ABOUT HERE]

The quantitative matching of Bond-CDS spreads, which is reported in Panel A in Table 3, is reasonably good for 5-year bonds. Overall, on the state-dependent dimension, the matching on the state G is close, but the model undershoots the data counterpart in state

[21] Because the CDS market is a zero-net-supply derivative market, how the secondary market liquidity of CDS market affects the pricing of CDS depends on market details. Bongaerts, De Jong, and Driessen (2011) show that indeed, the sellers of CDS contract earn a liquidity premium.

B. Cross-sectional wise, the model implied slope of Bond-CDS spread across ratings is a bit flatter than the data counterparts. For instance, in state B, the Bond-CDS spread goes from 76 bps for Aaa/Aa-rated bonds to 227 bps for Ba-rated bonds, while the model can explain about 55% of variation (from 74.5 bps of Aaa/Aa to 160 bps of Ba).

Overall, our model delivers the right magnitude for the empirically observed Bond-CDS spreads. The matching across macroeconomic states performs better than the matching across four rating categories, in that our model seems to produce too little variation ranging from superior grade bonds (Aaa/Aa) to speculative grade bonds (Ba). One possible avenue to explore which may help generate rating-dependent liquidity is that bonds with different ratings should have different clienteles, and we await future research in this interesting topic.

10-year Bond-CDS spreads and term structure Moving on to 10-year bonds, the model is doing a fair job in quantitatively matching the observed Bond-CDS spread in state B, while the performance in state G is poor. More importantly, similar to the discussion at the end of Section 4.3, one area our model clearly fails is to replicate a slightly downward sloping Bond-CDS term structure in the data. In our data from 2005 to 2012, the 5-year Bond-CDS spread is slightly higher than the 10-year counterpart across all rating categories, subject to the caveat that the difference may not be statistically significant taking standard deviations into account. It is worth noting that this downward sloping term structure in Bond-CDS spreads is inconsistent with the robust finding of longer-maturity bonds being less liquid documented in the empirical literature (e.g., Edwards, Harris, and Piwowar (2007); Bao, Pan, and Wang (2011)). In fact, Longstaff, Mithal, and Neis (2005) report a positive relation between Bond-CDS spread and maturity in their sample.

From the theoretical perspective, the model implied term structure for Bond-CDS spreads is upward sloping for investment grade bonds, but may turn downward sloping or flat for firms that are close to default.[22] However, the leverages in our Bond-CDS firm sample are

[22] The reason is similar to the inverted term structure of credit spreads for bonds with lower distance-to-default in standard default-driven models. For bonds that are close to default, the bond's stated maturity matters little, thus 5-year and 10-year bonds face similar illiquidity. Thus, the illiquidity discount per year (which is stated maturity) is higher for 5-year bonds, leading to downward sloping curve for Bond-CDS spreads for risky bonds.

relatively low which leads to upward-sloping Bond-CDS spreads for all ratings.

One possible explanation for the downward sloping Bond-CDS spreads over maturity, which is outside our model, is that the CDS spreads at different maturities are affected by liquidity differently. It is well recognized that CDS contracts are most liquid at the 5-year horizon when measured by the number of dealers offering quotes. If dealers are mainly selling CDS protections to regular investors and they possess market power (consistent with the empirical evidence in Bongaerts, De Jong, and Driessen (2011)), then the price of 10-year CDS contracts that are only offered by a small number of dealers tend to be higher than the price of 5-year CDS contracts with more competitive dealers. This may contribute to the relatively lower 10-year Bond-CDS spreads.

4.4.2 Bid-Ask Spread

Now we move on to the bid-ask spread as the second measure of non-default component. Recall that on the data side we combine both Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) to obtain the estimates of bid-ask spreads for corporate bonds, both across credit ratings and over business cycle. On the model side, again we rely on the empirical leverage distribution in Compustat firms across ratings and aggregate states to calculate the average of model implied bid-ask spreads. Since the average maturity in TRACE data is around 8 years, the model implied bid-ask spread is calculated as the weighted average between the bid-ask spread of a 5-year bond and a 10-year bond.

The model implied bid-ask spreads are reported in Panel B of Table 3, together with their empirical counterparts. The model is able to generate both cross-sectional and state-dependent patterns that quantitatively match what we observe in the data, especially in normal time. As mentioned before, we calibrate two state-dependent holding cost parameters (χ_G and χ_B) to match the bid-ask spread of investment grade bonds over macroeconomic states. Thus, it is less surprising that we are able to match the state-dependent pattern that bid-ask spreads more than double when the economy switches from state G to state B. However, in our model the bond's secondary market liquidity is endogenously linked to the firm's distance-to-default, which allows us to deliver the cross-sectional matching across three ratings. In normal times, the average bid-ask spread is 38 bps for superior grade bonds, 50 bps for investment grade bonds, and 91 bps for junk grade bonds, which are close to the data row in Table 3 taken from Edwards, Harris, and Piwowar (2007). The quantitative matching during recession is also satisfactory. Although not reported here, the model-implied bid-ask spread of longer-maturity bonds are higher than that of shorter-maturity bonds, which is consistent with previous empirical studies (eg. Edwards, Harris, and Piwowar (2007); Bao, Pan, and Wang (2011)). Finally, the implied bid-ask spread for the case of $\chi = 0$ is zero by definition (unreported in Table 3).

4.5 What if Pre-default Secondary Market is Perfectly Liquid?

Compared to earlier credit risk models that also incorporate macroeconomic risks, such as Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010), and Chen (2010), our model adds an illiquid secondary market for corporate bonds. By setting either the holding cost for type L investors, or the liquidity shock intensity, to zero (i.e., either $\chi_s = 0$ or ξ_s=0), we see what our model calibration implies about default risk and credit spreads in the absence of liquidity frictions, which helps isolate the effects of pre-default secondary market illiquidity.

The results are reported in the rows "$\chi = 0$" in Table 2 and Table 3. With $\chi_s = 0$, the credit spreads become significantly lower. For Aaa/Aa rated firms, in state G the 5-year spread falls from 72.9 bps to 10.1 bps (compared to the average spread of 55.7 bps in the data), while in state B it falls from 99 bps to 12.9 bps (compared to 107 bps in the data). Credit spreads for low-rated firms also fall, but by less in relative terms compared to highly-rated firms. Though not reported in Table 3, setting $\chi_s = 0$ generates model implied Bond-CDS spreads that are close to zero.[23]

Besides the credit spreads, shutting off the pre-default secondary market illiquidity lowers default probabilities as well. Quantity wise, the reduction for 5-year bonds with high rating is about 20 30% lower, while it is about 17% for speculative grade bonds. It is because

[23]They are not exactly zero due to the different methods in which we calculate bond yields and CDS spreads. For instance, for long-term (say 20 years) bonds that are close to default, credit spreads–which effectively assuming default risk is averaged out in 20 years–tend to be much lower than CDS spreads.

the secondary market illiquidity raises the rollover risk for firms, which in turn raises the probability of default. More importantly, this result illustrates that in order to obtain a precise decomposition of the default and liquidity components in credit spreads, we need to take into account the interactions between default risk and liquidity risk. We propose such a decomposition in the next section.

5. Structural Default-Liquidity Decomposition

Our structural model of corporate bonds features a full interaction between default and liquidity in determining the credit spreads of corporate bonds. It has been a common practice in the empirical literature to decompose the credit spread into liquidity and default components in an additive way, such as in Longstaff, Mithal, and Neis (2005). From the perspective of our model, this "intuitively appealing" decomposition tends to over-simplify the role of liquidity in determining the credit spread. More importantly, the additive structure often leads to a somewhat misguided interpretation that liquidity or default is the cause of the corresponding component, and each component would be the resulting credit spread if we were to shut down the other channel.

This interpretation may give rise to misleading answers in certain policy related questions. For instance, as our decomposition indicates, part of the default risk comes from the illiquidity in the secondary market. Thus, when the government is considering providing liquidity to the market, besides the direct effect on the credit spread by improving liquidity, there is also an indirect effect in lowering the default risk via the rollover channel and the firm's endogenous default decision. The traditional perspective with an additive structure often overlooks this indirect effect, a quantitatively important effect according to our study.

5.1 Decomposition Scheme

We propose a more detailed structural decomposition, which nests the additive default-liquidity decomposition common in the literature. Specifically, we further decompose the default part into the pure-default and liquidity-driven-default parts, and similarly decompose the liquidity part into the pure-liquidity and default-driven-liquidity parts:

$$\hat{cs} = \overbrace{\hat{cs}_{pureDEF} + \hat{cs}_{LIQ\to DEF}}^{\text{Default Component } \hat{cs}_{DEF}} + \overbrace{\hat{cs}_{pureLIQ} + \hat{cs}_{DEF\to LIQ}}^{\text{Liquidity Component} \hat{cs}_{LIQ}} \tag{19}$$

This way, we separate *causes* from *consequences*, and emphasize that lower liquidity (higher default) risk can lead to a rise in the credit spread via the default (liquidity) channel. Recognizing and further quantifying this endogenous interaction between liquidity and default is important in evaluating the economic consequence of policies that are either improving market liquidity (e.g., Term Auction Facilities or discount window loans) or alleviating default issues (e.g, direct bailouts).

Start with the default component. Imagine a hypothetical investor who is not subject to liquidity frictions, both pre- and post- default, and consider the spread that this investor demands over the risk-free rate. The resulting spread, denoted by $\hat{cs}_{DEF}$, only prices the default event given default threshold y_{def}, in line with Longstaff, Mithal, and Neis (2005) who use information from the relatively liquid CDS market to back out the default premium. Importantly, the default boundaries y^s_{def}'s in calculating $\hat{cs}_{DEF}$ remains the same given liquidity frictions as derived in equation (14).[24]

In contrast, we define the "Pure-Default" component $\hat{cs}_{pureDEF}$ as the spread implied by the benchmark Leland model without secondary market liquidity frictions at all (e.g., setting $\xi = 0$ or $\chi = 0$ for both pre- and post-default). Because the liquidity of the bond market leads to less rollover losses, equity holders default less often, i.e., $y^{Leland,s}_{def} < y^s_{def}$, where $y^{Leland,s}_{def}$ denotes the endogenous default boundary in Leland and Toft (1996) with time-varying aggregate states. The distinction between default boundaries implies a smaller pure-default component $\hat{cs}_{pureDEF}$ than the default component $\hat{cs}_{DEF}$. The difference $\hat{cs}_{DEF} - \hat{cs}_{pureDEF}$ gives the novel "Liquidity-Driven Default" component, which quantifies the effect that the illiquidity of secondary bond markets makes default more likely.

Now we move on the liquidity side. The liquidity component, in line with Longstaff, Mithal, and Neis (2005), is defined as $\hat{cs}_{LIQ} \equiv \hat{cs} - \hat{cs}_{DEF}$. That is to say, the liquidity component is

[24]Hypothetically, this is the situation where all other bond investors are still facing liquidity frictions as modeled. Hence, equity holders' default decision is not be affected.

the difference between the credit spread $\widehat{cs}$ implied by our model, and that required by a hypothetical investor without liquidity frictions, i.e., the spread $\hat{cs}_{DEF}$. Following a similar treatment to the default component, we further decompose $\hat{cs}_{LIQ}$ into a "Pure-Liquidity" component and a "Default-driven Liquidity" component. Let $\hat{cs}_{pureLIQ}$ be the spread of a bond that is subject to liquidity frictions as in Duffie, Gârleanu, and Pedersen (2005) but does not feature any default risk; this is the spread implied by our model as $P^s = p$ in 5 so that the bond price never drops (hence default free). The residual $\hat{cs}_{LIQ} - \hat{cs}_{pureLIQ}$ is what we term the default-driven liquidity part of our credit spread. Economically, when distance-to-default falls, lower bond prices give rise to a higher holding cost, which contributes to the default-driven liquidity part.

5.2 Default-Liquidity Decomposition

5.2.1 Level of credit spreads

We perform the above default-liquidity decomposition for 5-year bonds. We follow the same procedure as in Section 4.3.1, i.e., we first identify the cash-flow state at the firm-quarter level based on empirical leverage observed in Compustat, then aggregate them over firms and quarters.

The decomposition results for both aggregate states are presented in Panel I in Table 4. For each component, we report its absolute level in bps, as well as the percentage contribution to the credit spread. As expected, "pure default" component not only increases during recessions, but also rises for lower rating bonds. The fraction of credit spreads that can be explained by the "pure default" component starts from only 13% for Aaa/Aa rated bonds, and monotonically increases to about 50% for Ba rated bonds. Not surprisingly, the "pure liquidity" component is higher in state B (61 bps) than state G (46 bps), but do not vary across ratings. This is because the "pure liquidity" component captures the liquidity premium for a security whose holding cost is independent of ratings but higher in state B.

The remaining part of the observed credit spreads, which is around 25%$\sim$40% depending on the rating, can be attributed to the novel interaction terms, either "liquidity-driven default" or "default-driven liquidity". The "liquidity-driven default" part captures how corporate endogenous default decisions are affected by secondary market liquidity frictions via the rollover channel, which is quantitatively small for the highest rating firms (about 4% for Aaa/Aa rated bonds). As expected, its quantitative importance rises for low rating bonds: for Ba rated bonds, the liquidity-driven default accounts for about 13% of observed credit spreads.

The second interaction term, i.e., the "default-driven liquidity" component, captures how secondary market liquidity endogenously worsens when a bond is closer to default. Given a more illiquid secondary market for defaulted bonds, a lower distance-to-default leads to a worse secondary market liquidity because of the increased holding cost in 5. Different from "liquidity-driven default," the "default-driven liquidity" component is significant across all ratings: it accounts for about 20%∼27% of the credit spread when we move from Aaa/Aa ratings to Ba ratings.

Comparison to Longstaff, Mithal, and Neis (2005) How do our decomposition results compare to those documented in Longstaff, Mithal, and Neis (2005)? Based on CDS spreads and their structural model, Longstaff, Mithal, and Neis (2005) estimate that for 5-year Aaa/Aa rated bonds, the default component is about 51% of their credit spreads. For lower ratings, they report 56% for A, 71% for Baa, and 83% for Ba.

Overall, our decomposition in Table 4 gives a much lower default component compared to Longstaff, Mithal, and Neis (2005). More specifically, by adding up the "pure-default" and "liquidity-driven default" components, we have a default component of about 17% for Aaa/Aa rated bonds, 33% for A, 50% for Baa, and 64% for Ba. The driving force for this discrepancy is that we calibrate our model to match a much lower empirical ratio between CDS spread and credit spreads, especially for investment grade bonds. More specifically, in our sample, Table 2 and Table 3 imply that the ratio between CDS spread to credit spread is only about 51% for 5-year Baa rated bonds, a much lower number compared to 74% reported in Longstaff, Mithal, and Neis (2005).[25] A lower targeted CDS spread implies a smaller default component,

[25] In Longstaff, Mithal, and Neis (2005) whose sample period is from March 2001 to October 2002, the CDS spreads for Aaa/Aa rated bonds are about 59% of their corresponding credit spreads; 60% for A, 74%

which leads to a calibrated model with a relatively high liquidity component.

5.2.2 The change of credit spreads over aggregate states

This subsection focuses on a long-standing question that has interested empirical researchers, e.g., Dick-Nielsen, Feldhütter, and Lando (2012) and Friewald, Jankowitsch, and Subrahmanyam (2012a): How much of the soaring credit spread when the economy switches from boom to recession is due to increased credit risk, and how much is due to worsened secondary market liquidity? Our novel default-liquidity decomposition in (19) acknowledges that both liquidity and default risks for corporate bonds are endogenous and may affect each other. Given this feature, structural answers that rely on well-accepted economic structures are more appropriate than reduced-form approaches.

We report results in Table 4. As suggested by Panel II, increased default risks constitute a large fraction of the jump in credit spreads. The pure liquidity component is also quantitatively significant in explaining the rise of credit spreads: even for Ba rated bonds, about 13% of the rise when entering recessions is due to the lower secondary market illiquidity in state B.

When the economy encounters a recession, the higher default risk lowers secondary market liquidity further, giving rise to a greater "default-driven liquidity" part. Since worse liquidity in state B also pushes firms to default earlier, bond spreads rise because of a larger "liquidity-driven default" part. For low rated (say Ba) bonds, the "default-driven liquidity" channel (40%) is more important than that of "liquidity-driven default" (10%).

[TABLE 4 ABOUT HERE]

The interconnection between liquidity and default as documented above has important implications for the ongoing debate regarding how should accounting standards recognize credit losses on financial assets. The interesting interplay between liquidity and default and their respective accounting recognitions has been illustrated in the collapse of Asset-Backed-Securities market during the second half of 2007. As Acharya, Schnabl, and Suarez (2013)

for Baa, and 87% for Ba. In our data with a much longer sample period, these moments are 63% for Aaa/Aa, 52% for A, 51% for Baa, and 68% for B.

documented, as market participants are forward-looking, the liquidity problems (i.e., these conduits cannot roll over their short-term financing) occur before the actual credit-related losses (assets in the conduit start experiencing default). In a news release by Financial Accounting Standards Board (FASB) on 12/20/2013, the FASB Chairman Leslie F. Seidman noted that "the global financial crisis highlighted the need for improvements in the accounting for credit losses for loans and other debt instruments held as investment ... the FASB's proposed model would require more timely recognition of expected credit losses." However, there is no mentioning of the "liquidity" of these debt instrument at all. Our model not only suggests that (il)liquidity can affect the credit losses for these debt instruments, but more importantly offers a framework on how to evaluate the expected credit losses while taking into account the liquidity information.

5.3 Implications on Evaluating Liquidity Provision Policy

Our decomposition and its quantitative results are informative for evaluating the effect of policies that target lowering the borrowing cost of corporations in recession by injecting liquidity into the secondary market. As argued before, a full analysis of the effectiveness of such a policy must take account of how firms' default policies respond to liquidity conditions and how liquidity conditions respond to the default risks. These endogenous forces are what our model is aiming to capture.

Suppose that the government is committed to launching certain liquidity enhancing programs (e.g., Term Auction Facilities or discount window loans) whenever the economy falls into a recession, envisioning that the improved funding environment for financial intermediaries alleviates the worsening liquidity in the secondary bond market. Suppose that the policy is effective in making the secondary market in state B as liquid as that of state G. More precisely, the policy helps increase the meeting intensity between L investors and dealers in state B, so that λ_B rises from 20 to $\lambda_G = 50$; and reduce the state B holding cost parameter χ_B from 0.17 to $\chi_G = 0.12$.

In Table 5 we take the same cash flow levels for each rating class as in Table 4, and calculate the credit spreads with and without the state-B liquidity provision policy. We find that a state-B liquidity provision policy lowers state-B credit spreads by about 42 (236) bps for Aaa/Aa (Ba) rated bonds, which is about 51% (49%) of the corresponding credit spreads. Moreover, given the dynamic nature of our model, the state-B-only liquidity provision also affects firms' borrowing costs in state G: the state-G credit spreads for Aaa/Aa (Ba) rated bonds go down by 19 (129) bps, or about 30% (41%) of the corresponding credit spreads.

Our structural decomposition further allows us to investigate the underlying driving force for the effectiveness of this liquidity provision policy. By definition, the "pure default" component remains unchanged given any policy that only affects the secondary market liquidity.[26] In Table 5, we observe that the pure-liquidity component accounts for above 58% (56%) of the drop in spread for Aaa/Aa rated bonds in state G (state B). However, the quantitative importance of the pure-liquidity component goes down significantly when we walk down the rating spectrum: for Ba rated bonds, it only accounts for about 8∼10% of the decrease in the credit spread.

The market-wide liquidity provision not only reduces investors' required compensation for bearing liquidity risk, but also alleviates some default risk faced by bond investors. A better functioning financial market helps mitigate a firm's rollover risk and thus its default risk, and this force is captured by the "liquidity-driven default" part. The importance of this mechanism goes up for lower rated bonds (around 50%), but it remains quantitatively significant even for Aaa/Aa rated bonds (around 5%).

Given that the hypothetical policy was limited to only improving secondary market liquidity, the channel of "default-driven liquidity" is more intriguing, which only exists in our model with endogenous liquidity featuring a positive feedback loop between corporate default and secondary market liquidity. Not surprisingly, the contribution through "default-driven liquidity" is smaller; however, this interaction component is quantitatively significant: tt explains about 42% (35%) of policy effect for Ba (Aaa/Aa) rated bonds!

[TABLE 5 ABOUT HERE]

[26] The "pure default" component is defined by Leland and Toft (1996) which is independent of the secondary market liquidity.

6. Concluding Remarks

We build over-the-counter search frictions into a structural model of corporate bonds. In the model, firms default decisions interact with time varying macroeconomic and secondary market liquidity conditions. We calibrate the model to historical moments of default probability and empirical measures of liquidity. The model is able to match the observed credit spreads for corporate bonds with different rating classes, as well as various measures of non-default component studied in the previous literature. We propose a structural decomposition that captures the interaction of liquidity and default risks of corporate bonds over the business cycle and use this framework to evaluate the effects of liquidity provision policies during recessions. Our results identifies quantitatively important economic forces that were previously overlooked in empirical researches on corporate bonds.

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Appendix

A Additional Technical Details for the Model

1.1 Brownian Motion

The $\mathcal{Q}$-measure Brownian motion is given by

$$dZ_t^{\mathcal{Q}} = \frac{\sigma_m(s)}{\sqrt{\sigma_m^2(s_t)+\sigma_f^2}}dZ_t^m + \sqrt{1-\frac{\sigma_m^2(s)}{\sigma_m^2(s)+\sigma_f^2}}dZ_t^f + \frac{\sigma_m(s)}{\sigma(s)}\eta(s)\,dt,$$

1.2 Differential equations Debt & Equity

Throughout, $\text{diag}(\cdot)$ is the diagonalization operator mapping any row or column vector into a diagonal matrix (in which all off-diagonal elements are identically zero).

1.2.1 State Transition

As notational conventions, we use capitalized bold-faced letters (e.g., $\mathbf{X}$) to denote matrices, lower case bold face letters (e.g. $\mathbf{x}$) to denote vectors, and non-bold face letters denote scalars (e.g. x). The only exceptions are the value functions for debt and equity, $\mathbf{D}, \mathbf{E}$ respectively, which will be vectors, and the (diagonal) matrix of drifts, $\boldsymbol{\mu}$. Dimensions for most objects are given underneath the expression. While we focus on 2-aggregate-state case where $s \in \{G, B\}$, the Appendix presents general results for an arbitrary number of (Markov) aggregate states.

Given the aggregate state s, recall that we have assumed that the intensity of switching from state-H to state-L is ξ_s, and the L-state is absorbing, i.e., those L-type investors leave the market forever. However, an L-type bond holder meets a dealer with intensity λ_s and sells the bond for $B^s = \beta D_H^s + (1-\beta) D_L^s$ that he himself values at D_L^s. Then the L-type's *intensity-modulated surplus* when meeting the dealer can be rewritten as

$$\lambda_s (B^s - D_L^s) = \lambda_s \beta (D_H^s - D_L^s).$$

As a result, for the purpose of pricing, the *effective transitioning intensity* from L-type to H-type is $q_{Ls \to Hs} = \lambda_s \beta$ where λ_s is the state-dependent intermediation intensity and β is the investor's bargaining power.

1.2.2 Debt

Recall that the midpoint price is

$$P^s = \frac{A^s + B^s}{2} = \frac{(1+\beta) D_H^s + (1-\beta) D_L^s}{2} = w_H D_H^s + w_L D_L^s$$

with $w_H + w_L = 1$ and that holding costs are

$$hc^s = \chi_s [N - P^s] = \chi_s [N - w_H D_H^s - w_L D_L^s]$$

Stacking the holding cost function, we have

$$\mathbf{hc}(y) = \boldsymbol{\chi}[\mathbf{N} - \mathbf{W} \cdot \mathbf{D}(y)]$$

where $\boldsymbol{\chi} = \text{diag}\left([\chi_G, \chi_G, \chi_B, \chi_B]\right)$, $\mathbf{N} = [0, N, 0, N]^\top$ and

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ w_H & w_L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_H & w_L \end{bmatrix}$$

is the appropriate weighting matrix. Thus, $\mathbf{hc}(y)$ has identcially zero odd-numbered rows. Then, debt follows, on I_2,

$$\begin{aligned} \underbrace{\hat{\mathbf{R}} \cdot \mathbf{D}^{(2)}(y)}_{Discounting, 4\times 1} &= \underbrace{\boldsymbol{\mu}}_{4\times 4} \underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2} \underbrace{\boldsymbol{\Sigma}}_{4\times 4} \underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1} + \underbrace{\hat{\mathbf{Q}} \cdot \mathbf{D}^{(2)}(y)}_{Transition, 4\times 1} \\ &+ \underbrace{c\mathbf{1}_4}_{Coupon, 4\times 1} + \underbrace{m\left[p\mathbf{1}_4 - \mathbf{D}^{(2)}(y)\right]}_{Maturity, 4\times 1} - \underbrace{\mathbf{hc}(y)}_{Holding\,Cost, 4\times 1}, \end{aligned} \tag{20}$$

where $\hat{\mathbf{Q}}$ is the effective transition matrix (accounting for the OTC market outcome via the recovery intensity $\lambda_s \beta$) for debt out on interval I_2, given by

$$\hat{\mathbf{Q}} = \begin{bmatrix} -\xi_G - \zeta_G & \xi_G & \zeta_G & 0 \\ \beta\lambda_G & -\beta\lambda_G - \zeta_G & 0 & \zeta_G \\ \zeta_B & 0 & -\xi_B - \zeta_B & -\xi_B \\ 0 & \zeta_B & \beta\lambda_B & -\beta\lambda_B - \zeta_B \end{bmatrix},$$

and where

$$\hat{\mathbf{R}} \equiv \text{diag}\left([r_G, r_G, r_B, r_B]\right), \boldsymbol{\mu} \equiv \text{diag}\left([\mu_G, \mu_G, \mu_B, \mu_B]\right), \boldsymbol{\Sigma} \equiv \text{diag}\left(\left[\sigma_G^2, \sigma_G^2, \sigma_B^2, \sigma_B^2\right]\right).$$

Substituting it in, and noting that $\mathbf{X} = \mathbf{X}^{(n)}$ where n is the number of state, we have

$$\begin{aligned} \underbrace{\hat{\mathbf{R}}^{(2)} \cdot \mathbf{D}^{(2)}(y)}_{Discounting, 4\times 1} &= \underbrace{\boldsymbol{\mu}^{(2)}}_{4\times 4} \underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2} \underbrace{\boldsymbol{\Sigma}^{(2)}}_{4\times 4} \underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1} + \underbrace{\hat{\mathbf{Q}}^{(2)} \cdot \mathbf{D}^{(2)}(y)}_{Transition, 4\times 1} \\ &+ \underbrace{c\mathbf{1}_4}_{Coupon, 4\times 1} + \underbrace{m\left[p\mathbf{1}_4 - \mathbf{D}^{(2)}(y)\right]}_{Maturity, 4\times 1} - \underbrace{\left[\boldsymbol{\chi}^{(2)}\mathbf{N}^{(2)} - \boldsymbol{\chi}^{(2)}\mathbf{W} \cdot \mathbf{D}^{(2)}(y)\right]}_{Holding\,Cost, 4\times 1}, \end{aligned} \tag{21}$$

For simulation purposes, we need to incorporate $\mathbf{W}$ in such a way into $\mathbf{R}$ and $\mathbf{Q}$ that there is a unique discount rate in each state (i, s), i.e. $\mathbf{R}$ stays a diagonal discount matrix, and that $\mathbf{Q}$ stays a true transition matrix.[27] Thus, decompose $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ so that

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ w_H & -w_H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & w_H & -w_H \end{bmatrix}$$

First, we augement $\hat{\mathbf{Q}}$ to give the new transition matrix

$$\mathbf{Q} \equiv \hat{\mathbf{Q}} + \boldsymbol{\chi}\mathbf{W}_2 = \begin{bmatrix} -\xi_G - \zeta_G & \xi_G & \zeta_G & 0 \\ \beta\lambda_G + \chi_G w_H & -\beta\lambda_G - \zeta_G - \chi_G w_H & 0 & \zeta_G \\ \zeta_B & 0 & -\xi_B - \zeta_B & -\xi_B \\ 0 & \zeta_B & \beta\lambda_B + \chi_B w_H & -\beta\lambda_B - \zeta_B - \chi_B w_H \end{bmatrix}$$

Note that $\mathbf{Q}$ is still a **true** transition matrix. Next, let us define the diagonal discount matrix $\mathbf{R}^{(2)}$ as

$$\mathbf{R} \equiv \hat{\mathbf{R}} - \boldsymbol{\chi}\mathbf{W}_1 = \begin{bmatrix} r_G & 0 & 0 & 0 \\ 0 & r_G - \chi_G & 0 & 0 \\ 0 & 0 & r_B & 0 \\ 0 & 0 & 0 & r_B - \chi_B \end{bmatrix}$$

and we acknowledge the possibility of an indiviual entry being negative. Then, we have

$$\underbrace{\left[\mathbf{R}^{(2)} + m\mathbf{I}_4 - \mathbf{Q}^{(2)}\right]}_{4\times 4}\underbrace{\mathbf{D}^{(2)}(y)}_{4\times 1} = \underbrace{\boldsymbol{\mu}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)'(y)}_{4\times 1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma}^{(2)}}_{4\times 4}\underbrace{\left(\mathbf{D}^{(2)}\right)''(y)}_{4\times 1}$$
$$+ \underbrace{c\mathbf{1}_4}_{Coupon, 4\times 1} - \underbrace{\boldsymbol{\chi}^{(2)}\mathbf{N}^{(2)}}_{Holding\, Cost, 4\times 1} + \underbrace{m \cdot p\mathbf{1}_4}_{Maturity, 4\times 1}, \qquad (22)$$

the final form of the second-order ODE we are interested in.

Define $\mathbf{R}^{(1)} = \begin{bmatrix} r_G + \zeta_G & 0 \\ 0 & r_G - \chi_G + \zeta_G \end{bmatrix}$, $\mathbf{Q}^{(1)} \equiv \begin{bmatrix} -\xi_G & \xi_G \\ \beta\lambda_G + \chi_G w_H & -\beta\lambda_G - \chi_G w_H \end{bmatrix}$ and $\tilde{\mathbf{Q}}^{(1)} \equiv \begin{bmatrix} \zeta_G & 0 \\ 0 & \zeta_G \end{bmatrix} = \zeta_G \mathbf{I}_2$. On interval $I_1 = [y_{def}(G), y_{def}(B)]$, the bond is "dead" in state B, and the alive bonds

[27] Let $\mathbf{Q}$ be a transition matrix. Then define, for a finited time-interval Δt, the discrete-time transition matirx is given by

$$\mathbf{Q}_{discrete} \equiv \exp(\mathbf{Q}\Delta t)$$

Note that $\Delta t = 0$ implies that $\mathbf{Q}_{discrete} = \mathbf{I}$ and thus there is total persistence of the states. Also, note that since $\mathbf{Q}$ is a transition matrix that we have $\mathbf{Q}_{discrete}\mathbf{1} = \mathbf{1}$. This comes from the fact that

$$\exp(\mathbf{Q}\Delta t)\,\mathbf{1} = \mathbf{I}\cdot\mathbf{1} + \sum_{k=0}^{\infty} \frac{\mathbf{Q}^k \cdot \mathbf{1}\,(\Delta t)^k}{k!} = \mathbf{1}$$

as $\mathbf{Q}^k\mathbf{1} = \mathbf{0}$ for all k.

$\mathbf{D}^{(1)} = \left[D_H^{(G,1)}, D_L^{(G,1)}\right]^\top$ solve

$$\begin{aligned}\left[\mathbf{R}^{(1)} + m\mathbf{I}_2 - \mathbf{Q}^{(1)}\right]\mathbf{D}^{(1)} &= \boldsymbol{\mu}^{(1)}\left(\mathbf{D}^{(1)}\right)' + \frac{1}{2}\boldsymbol{\Sigma}^{(1)}\left(\mathbf{D}^{(1)}\right)'' + \tilde{\mathbf{Q}}^{(1)}\begin{bmatrix}\alpha_H^B \\ \alpha_L^B\end{bmatrix} v_U^B(y) \\ &+ \left(c\mathbf{1}_2 - \boldsymbol{\chi}^{(1)}\mathbf{N}^{(1)}\right) + m\cdot p\mathbf{1}_2 \end{aligned} \tag{23}$$

for

$$y \in I_1 = [y_{def}(G), y_{def}(B)],$$

where the last term is the recovery value in case of a jump to default brought about by a state jump.

The boundary conditions at $y = \infty$ and $y = y_{def}(G)$ are standard:

$$\lim_{y\to\infty}\left|\mathbf{D}^{(2)}(y)\right| < \infty, \text{ and } \mathbf{D}^{(1)}\left(y_{def}^G\right) = \begin{bmatrix}\alpha_H^G \\ \alpha_L^G\end{bmatrix} v_U^G\left(y_{def}^G\right) \tag{24}$$

For the boundary y_{def}^B, we must have value matching conditions for all functions across $y_{def}(B)$:

$$\mathbf{D}^{(2)}\left(y_{def}^B\right) = \begin{bmatrix}\mathbf{D}^{(1)}\left(y_{def}^B\right) \\ \begin{bmatrix}\alpha_H^B \\ \alpha_L^B\end{bmatrix} v_U^B\left(y_{def}^B\right)\end{bmatrix} \tag{25}$$

and smooth pasting conditions for functions that are alive across y_{def}^B ($\mathbf{x}_{[1:2]}$ selects the first 2 rows of vector $\mathbf{x}$):

$$\left(\mathbf{D}^{(2)}\right)'\left(y_{def}^B\right)_{[1:2]} = \left(\mathbf{D}^{(1)}\right)'\left(y_{def}^B\right). \tag{26}$$

1.2.3 Equity:

For equity, we have

$$\mathbf{RR} = \operatorname{diag}([r_G, r_B]), \boldsymbol{\mu\mu} = \operatorname{diag}([\mu_G, \mu_B]), \boldsymbol{\Sigma\Sigma} = \operatorname{diag}\left(\left[\sigma_G^2, \sigma_B^2\right]\right), \tag{27}$$

and

$$\mathbf{QQ}^{(2)} = \begin{bmatrix}-\zeta_G & \zeta_G \\ \zeta_B & -\zeta_B\end{bmatrix}$$

so that

$$\begin{aligned}\underbrace{\mathbf{RR}^{(2)}\mathbf{E}^{(2)}(y)}_{Discounting,2\times1} &= \underbrace{\boldsymbol{\mu\mu}^{(2)}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)'(y)}_{2\times1} + \frac{1}{2}\underbrace{\boldsymbol{\Sigma\Sigma}^{(2)}}_{2\times2}\underbrace{\left(\mathbf{E}^{(2)}\right)''(y)}_{2\times1} + \underbrace{\mathbf{QQ}^{(2)}\mathbf{E}^{(2)}(y)}_{Transition,2\times1} \\ &+ \underbrace{\mathbf{1}_2\exp(y)}_{Cashflow,2\times1} - \underbrace{(1-\pi)c\mathbf{1}_2}_{Coupon,2\times1} + \underbrace{m\left[\mathbf{S}^{(2)}\cdot\mathbf{D}^{(2)}(y) - p\mathbf{1}_2\right]}_{Rollover,2\times1}\end{aligned} \tag{28}$$

where $\mathbf{QQ}^{(2)}$ is the effective transition matrix for equity out on interval I_2 and $\mathbf{RR}^{(2)}$ is the effective discount matrix.

For I_1, define $\mathbf{RR}^{(1)} = r_G + \zeta_G, \boldsymbol{\mu\mu}^{(1)} = \mu_G, \boldsymbol{\Sigma\Sigma}^{(1)} = \sigma_G^2, \mathbf{QQ}^{(1)} = 0, \widetilde{\mathbf{QQ}}^{(1)} = \zeta_G$. Then

$$\left[\mathbf{RR}^{(1)} - \mathbf{QQ}^{(1)}\right] \mathbf{E}^{(1)}(y) = \boldsymbol{\mu\mu}^{(1)} \left(\mathbf{E}^{(1)}\right)'(y) + \frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(1)} \left(\mathbf{E}^{(1)}\right)''(y) + \widetilde{\mathbf{QQ}}^{(1)}\mathbf{0}_1 + \mathbf{1}_1 \exp(y) - (1-\pi) c\mathbf{1}_1 + m\left[\mathbf{S}^{(1)} \cdot \mathbf{D}^{(1)}(y) - p\mathbf{1}_1\right] \tag{29}$$

The particular solution is

$$\underbrace{\mathbf{E}^{(2)}(y)}_{2\times1} = \underbrace{\mathbf{GG}^{(2)}}_{2\times4} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma\Gamma}^{(2)} y\right)}_{4\times4} \cdot \underbrace{\mathbf{bb}^{(2)}}_{4\times1} + {}_{2\times8}\underbrace{\mathbf{KK}^{(2)}}_{8\times8} \underbrace{\exp\left(\boldsymbol{\Gamma}^{(2)} y\right)}_{4\times2} \underbrace{\mathbf{b}^{(2)}}_{} + {}_{2\times1}\underbrace{\mathbf{kk}_0^{(2)}}_{} + {}_{2\times1}\underbrace{\mathbf{kk}_1^{(2)}}_{} \exp(y) \text{ for} y \in I_2$$

$$\underbrace{\mathbf{E}^{(1)}(y)}_{1\times1} = \underbrace{\mathbf{GG}^{(1)}}_{1\times2} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma\Gamma}^{(1)} y\right)}_{2\times2} \cdot \underbrace{\mathbf{bb}^{(1)}}_{2\times1} + {}_{1\times4}\underbrace{\mathbf{KK}^{(1)}}_{} \underbrace{\exp\left(\boldsymbol{\Gamma}^{(1)} y\right)}_{4\times4} \underbrace{\mathbf{b}^{(1)}}_{4\times1} + \underbrace{\mathbf{kk}_0^{(1)}}_{1\times1} + \underbrace{\mathbf{kk}_1^{(1)}}_{1\times1} \exp(y) \text{ for } y \in I_1$$

where $\mathbf{GG}^{(i)}$,$\boldsymbol{\Gamma\Gamma}^{(i)}$, $\mathbf{bb}^{(i)}$,$\mathbf{KK}^{(i)}$, $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$ for $i \in \{1,2\}$ are given below. In particular, the constant vector $\mathbf{bb}^{(i)}$ is determined by boundary conditions similar to those for debt.

1.2.4 Defaulted bonds

Consider now a defaulted bond. Its value $D_l^i(y)$ will be determined by the holding cost $hc_{def}^s(y) = \chi_s^{def}\left[N^{def} - P_s^{def}(y)\right]$. Let $\mathbf{V}(y) = \text{diag}\left(\left[v_U^G(y), v_U^G(y), v_U^B(y), v_U^B(y)\right]\right)$. Then, we can write out

$$\hat{\mathbf{R}} \cdot \mathbf{D}^{def}(y) = \theta\left[\mathbf{V}(y)\boldsymbol{\alpha} - \mathbf{D}^{def}\right] + \hat{\mathbf{Q}}^{def} \cdot \mathbf{D}^{def}(y) - \boldsymbol{\chi}^{def}\left[\mathbf{N}^{def} - \mathbf{W}^{def}\mathbf{D}^{def}(y)\right] \tag{30}$$

which is solved by

$$\mathbf{D}^{def}(y) = \left[\hat{\mathbf{R}} + \theta\mathbf{I}_n - \hat{\mathbf{Q}}^{def} - \boldsymbol{\chi}^{def}\mathbf{W}^{def}\right]^{-1}\left[\theta\mathbf{V}(y)\boldsymbol{\alpha} - \boldsymbol{\chi}^{def}\mathbf{N}^{def}\right]$$

Suppose now that $\mathbf{N}^{def} = \mathbf{V}(y)\mathbf{n}^{def}$. Then, we have

$$\mathbf{D}^{def}(y) = \left[\hat{\mathbf{R}} + \theta\mathbf{I}_n - \hat{\mathbf{Q}}^{def} - \boldsymbol{\chi}^{def}\mathbf{W}^{def}\right]^{-1}\mathbf{V}(y)\left[\theta\boldsymbol{\alpha} - \boldsymbol{\chi}^{def}\mathbf{n}^{def}\right] \tag{31}$$

where we used that fact that $\boldsymbol{\chi}^{def}\mathbf{V}(y) = \mathbf{V}(y)\boldsymbol{\chi}^{def}$ as both are diagonal matrices.

1.2.5 Generalization to n aggregate states

We follow the Markov-modulated dynamics approach of Jobert and Rogers (2006).

We note that there are multiple possible bankruptcy boundaries, $y_b(s)$, for each aggregate state s one boundary. Order states s such that $s > s'$ implies that $y_b(s) > y_b(s')$ and denote the intervals $I_s = [y_b(s), y_b(s+1)]$ where $y_b(n+1) = \infty$, so that $I_s \cap I_{s+1} = y_b(s+1)$. Finally, let $\boldsymbol{y}_b = [y_b(1), ..., y_b(n)]^\top$ be the vector of bankruptcy boundaries.

It is important to have a clean notational arrangement to handle the proliferation of states. Let $D_l^{(s)}$ denote the value of debt for a creditor in individual liquidity state l and with aggregate state s. We will use the following notation: $D_l^{(s,i)} \equiv D_l^{(s)}, y \in I_i$, that is $D_l^{(s,i)}$ is the restriction of $D_l^{(s)}$to the interval I_i. It is now clear that $D_l^{(s,i)} = recovery$ for any $i < s$, as it would imply that the company immediately defaults in interval I_i for state s. Let us, for future reference, call debt in states $i < s$ dead and in states $i \geq s$ alive. Finally, let us stack the alive functions along states s but still restricted to interval i so that $\mathbf{D}^{(i)} = \left[D_H^{(1,i)}, D_L^{(1,i)}, ..., D_H^{(i,i)}, D_L^{(i,i)}\right]^\top$ where $D_l^{(s,i)}$ has s denoting the state, i denotes the interval and l denotes the individual liquidity state. The separation of s and i will clarify the pasting arguments that apply when y crosses from one interval to the next. Let

$$\underbrace{\mathbf{I}_i}_{i\times i} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{32}$$

be the i-dimensional identity matrix, and let

$$\underbrace{\mathbf{1}_i}_{i\times 1} = [1, ..., 1]^\top \tag{33}$$

be a column vector of ones.

Fundamental parameters. Let $\hat{\mathbf{R}} = \text{diag}\left([r_1, r_1, ..., r_n, r_n]\right)$ and let $\mathbf{RR} = \text{diag}\left([r_1, ..., r_n]\right)$. Let $\hat{\mathbf{Q}}$ summarize the OTC-augmented transition intensities. Define the building block matrices $\mathbf{B}_s^{W_1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B}_s^{W_2} = \begin{bmatrix} 0 & 0 \\ w_H & -w_H \end{bmatrix}$ so that $\mathbf{B}_s^{W} = \begin{bmatrix} 0 & 0 \\ w_H & w_L \end{bmatrix} = \mathbf{B}_s^{W_1} + \mathbf{B}_s^{W_2}$. Then, we have

$$\mathbf{W}_1 = \text{diag}\left(\left[\mathbf{B}_1^{W_1}, ..., \mathbf{B}_n^{W_1}\right]\right), \mathbf{W}_2 = \text{diag}\left(\left[\mathbf{B}_1^{W_2}, ..., \mathbf{B}_n^{W_2}\right]\right), \boldsymbol{\chi} = \text{diag}\left([\chi_1, \chi_1, ..., \chi_n, \chi_n]\right)$$

and we define the diagonal discount matrix $\mathbf{R}$ and the transition matrix $\mathbf{Q}$ as

$$\mathbf{R} = \hat{\mathbf{R}} - \boldsymbol{\chi}\mathbf{W}_1, \mathbf{Q} = \hat{\mathbf{Q}} + \boldsymbol{\chi}\mathbf{W}_2$$

Let $\boldsymbol{\mu} = \text{diag}\left([\mu_1, \mu_1, ..., \mu_n, \mu_n]\right)$, $\boldsymbol{\Sigma} = \text{diag}\left(\left[\sigma_1^2, \sigma_1^2, ..., \sigma_n^2, \sigma_n^2\right]\right)$.

Define $\mathbf{X}^{(n)} = \mathbf{X}$, that is the restriction to the set $I_n = [y_b(n), \infty)$ does not change any of the matrices. Let $\mathbf{Q}^{(i)}$ be the transition matrix of jumping into an alive state $s' \leq i$ when $y \in I_i$ and in an alive state $s \leq i$. Let $\tilde{\mathbf{Q}}^{(i)}$ be the transition matrix of jumping into a default state $s' > i$ when $y \in I_i$ and in an alive state $s \leq i$, so that $\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^{(n)} = 0$. The decomposition for $y \in I_i$ takes the following form

$$\begin{aligned}
\underbrace{\mathbf{Q}_{[1:2i]}}_{2i\times 2n} &= \left[\underbrace{\mathbf{Q}^{(i)} - \text{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right)}_{2i\times 2i} \middle| \underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)}\right] \\
\underbrace{\mathbf{R}_{[1:2i]}}_{2i\times 2n} &= \left[\underbrace{\mathbf{R}^{(i)} - \text{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right)}_{2i\times 2i} \middle| \underbrace{\mathbf{0}_{2i\times 2(n-i)}}_{2i\times 2(n-i)}\right]
\end{aligned}$$

so that $\mathbf{Q}^{(i)}, \tilde{\mathbf{Q}}^{(i)}, \mathbf{R}^{(i)}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma}^{(i)}$ are given by

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(i)} &= \mathbf{Q}_{[1:2i,2i+1:2n]} \\
\mathbf{Q}^{(i)} &= \mathbf{Q}_{[1:2i,1:2i]} + \text{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right) \\
\mathbf{R}^{(i)} &= \mathbf{R}_{[1:2i,1:2i]} + \text{diag}\left(\tilde{\mathbf{Q}}^{(i)}\mathbf{1}_{2(n-i)}\right) \\
\boldsymbol{\mu}^{(i)} &= \boldsymbol{\mu}_{[1:2i,1:2i]} \\
\boldsymbol{\Sigma}^{(i)} &= \boldsymbol{\Sigma}_{[1:2i,1:2i]}
\end{aligned}$$

where $\mathbf{X}_{[1:k]}$ takes the first k rows of matrix $\mathbf{X}$ and $\mathbf{X}_{[1:k,1:l]}$ takes the first k rows and l columns of matrix $\mathbf{X}$. Note that in the $n = 2$ aggregate state case, we have $\tilde{\mathbf{Q}}^{(1)} = \text{diag}\left[\tilde{\mathbf{Q}}^{(1)}\mathbf{1}_2\right]$ as we have no joint jumps in liquidity state and in the aggregate state.

Let $\mathbf{v}^{(i)}\exp(y)$ be the recovery or salvage value of the firm when default is declared in states $s>i$ for $y\in I_i$. Thus, $\mathbf{v}^{(i)}$ is a vector containing recovery values for states $(i+1,...,n)\times(H,L)$ (i.e., it is of dimension $2(n-i)\times 1$).

Debt valuation within an interval I_i. Debt valuation follows the following differential equation on interval I_i:

$$\begin{aligned}\left(\mathbf{R}^{(i)}+m\mathbf{I}_i-\mathbf{Q}^{(i)}\right)\mathbf{D}^{(i)} &= \boldsymbol{\mu}^{(i)}\left(\mathbf{D}^{(i)}\right)'+\frac{1}{2}\boldsymbol{\Sigma}^{(i)}\left(\mathbf{D}^{(i)}\right)''+1_{\{i<n\}}\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y)\\ &\quad+\left(c\mathbf{1}_{2i}-\boldsymbol{\chi}^{(i)}\mathbf{N}^{(i)}\right)+m\cdot p\mathbf{1}_{2i} \qquad (34)\end{aligned}$$

where $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y)$ represents the intensity of jumping into default times the recovery in the default state and $m\cdot p\mathbf{1}_{2i}$ represents the intensity of randomly maturing times the payoff in the maturity state. Next, let us conjecture a solution of the kind $\mathbf{g}\exp(\gamma y)+\mathbf{k}_0^{(i)}+\mathbf{k}_1^{(i)}\exp(y)$ where $\mathbf{g}$ is a vector and γ is a scalar. The particular part stemming from $\mathbf{c}^{(i)}$ is solved by a term $\mathbf{k}_0^{(i)}$ with

$$\underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1}=\underbrace{\left(\mathbf{R}^{(i)}+m\mathbf{I}_i-\mathbf{Q}^{(i)}\right)^{-1}}_{2i\times 2i}\underbrace{(c+m\cdot p)\mathbf{1}_{2i}-\boldsymbol{\chi}^{(i)}\mathbf{N}^{(i)}}_{2i\times 1} \qquad (35)$$

and the particular part stemming from $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}$ is solved by a term $\mathbf{k}_1^{(i)}\exp(y)$ with

$$\underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1}=\underbrace{\left(\mathbf{R}^{(i)}+m\mathbf{I}_i-\mathbf{Q}^{(i)}-\boldsymbol{\mu}^{(i)}-\frac{1}{2}\boldsymbol{\Sigma}^{(i)}\right)^{-1}}_{2i\times 2i}\underbrace{\tilde{\mathbf{Q}}^{(i)}}_{2i\times 2(n-i)}\underbrace{\mathbf{v}^{(i)}}_{2(n-i)\times 1} \qquad (36)$$

It should be clear that $\mathbf{k}_1^{(n)}=\mathbf{0}$ as on I_n there is no jump in the aggregate state that would result in immediate default. Plugging in, dropping the $\mathbf{c}^{(i)}$ and $\tilde{\mathbf{Q}}^{(i)}\mathbf{v}^{(i)}\exp(y)$ terms, canceling out $\exp(\gamma y)>0$, we have

$$\mathbf{0}_{2i}=\left(\mathbf{Q}^{(i)}+m\mathbf{I}_i-\mathbf{R}^{(i)}\right)\mathbf{g}+\boldsymbol{\mu}^{(i)}\gamma\mathbf{g}+\frac{1}{2}\boldsymbol{\Sigma}^{(i)}\gamma^2\mathbf{g} \qquad (37)$$

Following JR06, we premultiply by $2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}$ and define $\mathbf{h}=\gamma\mathbf{g}$ to get

$$\gamma\mathbf{g} = \mathbf{h} \qquad (38)$$

$$\gamma\mathbf{h} = -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)}\mathbf{h}+2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)}+m\mathbf{I}_i-\mathbf{Q}^{(i)}\right)\mathbf{g} \qquad (39)$$

Stacking the vectors $\mathbf{j}=\begin{bmatrix}\mathbf{g}\\ \mathbf{h}\end{bmatrix}$ we have

$$\gamma\mathbf{j}=\begin{bmatrix}\mathbf{0}_{2i} & \mathbf{I}_{2i}\\ 2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\left(\mathbf{R}^{(i)}+m\mathbf{I}_i-\mathbf{Q}^{(i)}\right) & -2\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu}^{(i)}\end{bmatrix}\mathbf{j}=\underbrace{\mathbf{A}^{(i)}}_{4i\times 4i}\mathbf{j} \qquad (40)$$

where $\mathbf{I}$ is of appropriate dimensions. The problem is now a simple eigenvalue-eigenvector problem and each solution j is a pair $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1},\underbrace{\mathbf{j}_j^{(i)}}_{4i\times 1}\right)$ (or rather $\left(\underbrace{\gamma_j^{(i)}}_{1\times 1},\underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1}\right)$, as the vector $\mathbf{j}_j^{(i)}$ contains the same information as $\mathbf{g}_j^{(i)}$ when we know $\gamma_j^{(i)}$, so we discard the lower half of $\mathbf{j}_j^{(i)}$). The number of solutions j to this eigenvector-eigenvalue problem is $4i$. Let

$$\mathbf{G}^{(i)} \equiv \left[\mathbf{g}_1^{(i)}, ..., \mathbf{g}_{2\times 2\times i}^{(i)}\right] \tag{41}$$

be the matrix of eigenvectors, and let

$$\boldsymbol{\gamma}^{(i)} \equiv \left[\gamma_1^{(i)}, ..., \gamma_{2\times 2\times i}^{(i)}\right]' \tag{42}$$

$$\boldsymbol{\Gamma}^{(i)} \equiv \text{diag}\left[\boldsymbol{\gamma}^{(i)}\right] \tag{43}$$

be the corresponding vector and diagonal matrix, respectively, of eigenvalues.

The general solution on interval i is thus

$$\underbrace{\mathbf{D}^{(i)}}_{2i\times 1} = \underbrace{\mathbf{G}^{(i)}}_{2i\times 4i} \cdot \underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)} y\right)}_{4i\times 4i} \cdot \underbrace{\mathbf{c}^{(i)}}_{4i\times 1} + \underbrace{\mathbf{k}_0^{(i)}}_{2i\times 1} + \underbrace{\mathbf{k}_1^{(i)}}_{2i\times 1} \exp(y) \tag{44}$$

where the constants $\mathbf{c}^{(i)} = \left[c_1^{(i)}, ..., c_{4i}^{(i)}\right]^\top$ will have to be determined via conditions at the boundaries of interval I_i (**NOTE:** $c_j^{(i)} \neq c$ where c is the coupon payment).

Boundary conditions. The different value functions $\mathbf{D}^{(i)}$ for $i \in \{1, ..., n\}$are linked at the boundaries of their domains I_i. Note that $I_i \cap I_{i+1} = \{y_B(i+1)\}$ for $i < n$.

For $i = n$, we can immediately rule out all positive solutions to γ as debt has to be finite and bounded as $y \to \infty$, so that the entries of $\mathbf{C}^{(n)}$ corresponding to positive eigenvalues will be zero:[28]

$$\lim_{y\to\infty} \left|\mathbf{D}^{(n)}(y)\right| < \infty \tag{45}$$

For $i < n$, we must have value matching of the value functions that are alive across the boundary, and we must have value matching of the value functions that die across the boundary:

$$\mathbf{D}^{(i+1)}(y_B(i+1)) = \begin{bmatrix} \mathbf{D}^{(i)}(y_B(i+1)) \\ \begin{bmatrix} v_H^{i+1} \\ v_L^{i+1} \end{bmatrix} \exp(y_B(i+1)) \end{bmatrix} \tag{46}$$

For $i < n$, we must have mechanical (i.e. non-optimal) smooth pasting of the value functions that are alive across the boundary:

$$\left(\mathbf{D}^{(i+1)}\right)'(y_B(i+1))_{[1:2i]} = \left(\mathbf{D}^{(i)}\right)'(y_B(i+1)) \tag{47}$$

where $\mathbf{x}_{[1...2i]}$ selects the first $2i$ rows of vector $\mathbf{x}$.

Lastly, for $i = 1$, we must have

$$\mathbf{D}^{(1)}(y_B(1)) = \begin{bmatrix} v_H^1 \\ v_L^1 \end{bmatrix} \exp(y_B(1)) \tag{48}$$

Full solution. We can now state the full solution to the debt valuation given cut-off strategies:

[28]According to JR06, there are exactly $2 \times |S| = 2n$ eigenvalues of $\mathbf{A}$ in the left open half plane (i.e. negative) and $2n$ eigenvalues in the right open half plane (i.e. positive) (actually, they only argue that this holds if $\boldsymbol{\mu} = \mathbf{R} - \frac{1}{2}\boldsymbol{\Sigma}$, but maybe not for general $\boldsymbol{\mu}$).

Proposition 4. *The debt value functions* $\mathbf{D}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{D}(y)=\begin{cases}\underbrace{\mathbf{D}^{(n)}(y)}_{2n\times 1}=\mathbf{G}^{(n)}\cdot\exp\left(\mathbf{\Gamma}^{(n)}y\right)\cdot\mathbf{c}^{(n)}+\mathbf{k}_0^{(n)} & y\in I_n\\ \vdots & \vdots\\ \underbrace{\mathbf{D}^{(i)}(y)}_{2i\times 1}=\mathbf{G}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{c}^{(i)}+\mathbf{k}_0^{(i)}+\mathbf{k}_1^{(i)}\exp(y) & y\in I_i\\ \vdots & \vdots\\ \underbrace{\mathbf{D}^{(1)}(y)}_{2\times 1}=\mathbf{G}^{(1)}\cdot\exp\left(\mathbf{\Gamma}^{(1)}y\right)\cdot\mathbf{c}^{(1)}+\mathbf{k}_0^{(1)}+\mathbf{k}_1^{(1)}\exp(y) & y\in I_1\end{cases}$$

with the following boundary conditions to pin down vectors $\mathbf{c}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{D}^{(n)}(y)}_{2n\times 1}\right| < \infty \tag{49}$$

$$\underbrace{\mathbf{D}^{(i+1)}(y_B(i+1))}_{2(i+1)\times 1} = \underbrace{\begin{bmatrix}\mathbf{D}^{(i)}(y_B(i+1))\\ \begin{bmatrix}v_H^{i+1}\\ v_L^{i+1}\end{bmatrix}\exp(y_B(i+1))\end{bmatrix}}_{2(i+1)\times 1} \tag{50}$$

$$\underbrace{\left(\mathbf{D}^{(i+1)}\right)'(y_B(i+1))_{[1:2i]}}_{2i\times 1} = \underbrace{\left(\mathbf{D}^{(i)}\right)'(y_B(i+1))}_{2i\times 1} \tag{51}$$

$$\underbrace{\mathbf{D}^{(1)}(y_B(1))}_{2\times 1} = \underbrace{\begin{bmatrix}v_H^1\\ v_L^1\end{bmatrix}\exp(y_B(1))}_{2\times 1} \tag{52}$$

where $\mathbf{x}_{[1:2i]}$ *selects the first* $2i$ *rows of vector* $\mathbf{x}$*.*

.

Note that the derivative of the debt value vector is

$$\underbrace{\left(\mathbf{D}^{(i)}\right)'(y)}_{2i\times 1}=\mathbf{G}^{(i)}\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{c}^{(i)}+\mathbf{k}_1^{(i)}\exp(y) \tag{53}$$

where we note that $\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right)=\exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma}^{(i)}$as both are diagonal matrices (although this interchangeability only is important when $s=1$ as it then helps collapse some equations).

The first boundary condition (49) essentially implies that we can discard any positive entries of $\boldsymbol{\gamma}^{(n)}$ by setting the appropriate coefficients of $\mathbf{C}^{(n)}$ to 0. The second boundary condition (50) implies that we have value matching at any boundary $y_B(i+1)$ for $i<n$, be it to a continuation state or a bankruptcy state . The third boundary condition (51) implies that we also have smooth pasting at the boundary $y_B(i+1)$ for those states in which the firm stays alive on both sides of the boundary. Finally, the fourth boundary condition (52) implies value matching at the boundary $y_B(1)$, but of course only for those states in which the firm is still alive.

1.2.6 Equity

The equity holders are unaffected by the individual liquidity shocks the debt holders are exposed to. The only shocks the equity holders are directly exposed to are the shifts in $\mu(s)$ and $\sigma(s)$, i.e. shifts to the cash-flow process.

However, as debt has maturity and is rolled over, equity holders are indirectly affected by liquidity shocks in the market through the effect it has on debt prices. Thus, when debt matures, it is either rolled over if the debt holders are of type H, or it is reissued to different debt holders in the case that the former debt holder is of type L. Either way, there is a cash flow (inflow or outflow) of $m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y)-p\mathbf{1}_i\right]$ at each instant as a mass $m\cdot dt$ of debt holders matures on $[t,t+dt]$.

For notational ease, we will denote by double letters (e.g. $\mathbf{xx}$) a constant for equity that takes a similar place as a single letter (i.e. $\mathbf{x}$) constant for debt. Then, the HJB for equity on interval I_i is given by

$$\begin{aligned}\left(\mathbf{RR}^{(i)}-\mathbf{QQ}^{(i)}\right)\mathbf{E}^{(i)}(y) &= \boldsymbol{\mu\mu}^{(i)}\left(\mathbf{E}^{(i)}\right)'(y)+\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\left(\mathbf{E}^{(i)}\right)''(y)+1_{\{i<n\}}\underbrace{\widetilde{\mathbf{QQ}}^{(i)}\mathbf{0}_i}_{Default}\\ &\quad +\underbrace{\mathbf{1}_i\exp(y)}_{Cashflow}-\underbrace{(1-\pi)c\mathbf{1}_i}_{Coupon}+\underbrace{m\left[\mathbf{S}^{(i)}\cdot\mathbf{D}^{(i)}(y)-p\mathbf{1}_i\right]}_{Rollover}\end{aligned}\tag{54}$$

where

$$\widetilde{\mathbf{QQ}}^{(i)} = \mathbf{QQ}_{[1:i,i+1:n]}\tag{55}$$

$$\mathbf{QQ}^{(i)} = \mathbf{QQ}_{[1:i,1:i]}+\operatorname{diag}\left(\widetilde{\mathbf{QQ}}^{(i)}\mathbf{1}_{n-i}\right)\tag{56}$$

$$\mathbf{RR}^{(i)} = \mathbf{RR}_{[1:i,1:i]}+\operatorname{diag}\left(\widetilde{\mathbf{QQ}}^{(i)}\mathbf{1}_{n-i}\right)\tag{57}$$

$$\boldsymbol{\mu\mu}^{(i)} = \boldsymbol{\mu\mu}_{[1:i,1:i]}\tag{58}$$

$$\boldsymbol{\Sigma\Sigma}^{(i)} = \boldsymbol{\Sigma\Sigma}_{[1:i,1:i]}\tag{59}$$

Note that $\widetilde{\mathbf{QQ}}^{(i)}$ is the transition matrix only between aggregate states that is also an $i\times i$ square matrix, and $\mathbf{S}^{(i)}$ is a $i\times 2i$ matrix that selects which debt values the firm is able to issue (each row has to sum to 1), and m is a scalar (**NOTE**: In contrast to $\mathbf{R}$, the matrix $\mathbf{RR}$ does not contain the maturity intensity m). For example, for $i=2$, if the company is able to place debt only to H types, then $\mathbf{S}^{(2)}=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}$. It is important that for reach row i only entries $2i-1$ and $2i$ are possibly nonzero, whereas all other entries are identically zero (otherwise, one would issue bonds belonging to a different state).

Writing out $\mathbf{D}^{(i)}(y)=\mathbf{G}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}$and conjecturing a solution to the particular, non-constant part $\underbrace{\mathbf{KK}^{(i)}}_{i\times 4i}\underbrace{\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)}_{4i\times 4i}\underbrace{\mathbf{c}^{(i)}}_{4i\times 1}$, we have

$$\begin{aligned}&\left(\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}\right)\mathbf{KK}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}\\ =&\left[\boldsymbol{\mu\mu}^{(i)}\cdot\mathbf{KK}^{(i)}\cdot\boldsymbol{\Gamma}^{(i)}+\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\mathbf{KK}^{(i)}\cdot\left(\boldsymbol{\Gamma}^{(i)}\right)^2+m\cdot\mathbf{S}^{(i)}\cdot\mathbf{G}^{(i)}\right]\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}\end{aligned}\tag{60}$$

We can solve this by considering each $\gamma_j^{(i)}$ separately — recall that $\mathbf{c}^{(i)}$ is a vector and $\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)$ is a *diagonal* matrix and in total there are $4i$ different roots. Consider the part of the particular part $\mathbf{S}^{(i)}\cdot\mathbf{g}_j^{(i)}\exp\left(\gamma_j^{(i)}y\right)\cdot c_j^{(i)}$

and our conjecture gives $\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1}\underbrace{\exp\left(\gamma_j^{(i)}y\right)}_{1\times 1}\cdot\underbrace{c_j^{(i)}}_{1\times 1}$ for each root $j\in[1,...,4i]$. Plugging in and multiplying out the scalar $\exp\left(\gamma_j^{(i)}y\right)c_j^{(i)}$, we find that

$$\left(\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}\right)\mathbf{KK}_j^{(i)}=\boldsymbol{\mu\mu}^{(i)}\cdot\mathbf{KK}_j^{(i)}\cdot\gamma_j^{(i)}+\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\mathbf{KK}_j^{(i)}\cdot\left(\gamma_j^{(i)}\right)^2+m\cdot\mathbf{S}^{(i)}\cdot\mathbf{g}_j^{(i)} \tag{61}$$

Solving for $\mathbf{KK}_j^{(i)}$, we have

$$\underbrace{\mathbf{KK}_j^{(i)}}_{i\times 1}=\underbrace{\left[\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}-\boldsymbol{\mu\mu}^{(i)}\cdot\gamma_j^{(i)}-\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\cdot\left(\gamma_j^{(i)}\right)^2\right]^{-1}}_{i\times i}m\cdot\underbrace{\mathbf{S}^{(i)}}_{i\times 2i}\underbrace{\mathbf{g}_j^{(i)}}_{2i\times 1} \tag{62}$$

Finally, for the homogenous part we use the same approach as above, but now we have less states as the individual liquidity state drops out. Thus, we conjecture $\mathbf{gg}\exp(\gamma\gamma y)$ to get

$$\mathbf{0}_i=\left(\widetilde{\mathbf{QQ}}^{(i)}-\widetilde{\mathbf{RR}}^{(i)}\right)\mathbf{gg}+\boldsymbol{\mu\mu}^{(i)}\gamma\gamma\mathbf{gg}+\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\gamma\gamma\mathbf{gg} \tag{63}$$

so that, again, we have the following eigenvector eigenvalue problem

$$\gamma\gamma\mathbf{jj}=\begin{bmatrix}\mathbf{0}_i & \mathbf{I}_i\\ 2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1}\left(\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}\right) & -2\left(\boldsymbol{\Sigma\Sigma}^{(i)}\right)^{-1}\boldsymbol{\mu\mu}^{(i)}\end{bmatrix}\mathbf{jj}=\underbrace{\mathbf{AA}^{(i)}}_{2i\times 2i}\mathbf{jj} \tag{64}$$

which gives $\left(\gamma\gamma_j^{(i)},\mathbf{gg}_j^{(i)}\right)$ for $j\in[1,...,2i]$ solutions. We stack these into a matrix of eigenvectors $\mathbf{GG}^{(i)}$ and a vector of eigenvalues $\boldsymbol{\gamma\gamma}^{(i)}$, from which we define the diagonal matrix of eigenvalues $\boldsymbol{\Gamma\Gamma}^{(i)}\equiv\operatorname{diag}\left(\boldsymbol{\gamma\gamma}^{(i)}\right)$. What remains is to solve for $\mathbf{kk}_0^{(i)}$ and $\mathbf{kk}_1^{(i)}$. We have

$$\mathbf{kk}_0^{(i)}=\left[\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}\right]^{-1}\left[-(1-\pi)c\mathbf{1}_i+m\left(\mathbf{S}^{(i)}\mathbf{k}_0^{(i)}-p\mathbf{1}_i\right)\right] \tag{65}$$

and

$$\mathbf{kk}_1^{(i)}=\left[\widetilde{\mathbf{RR}}^{(i)}-\widetilde{\mathbf{QQ}}^{(i)}-\boldsymbol{\mu\mu}^{(i)}-\frac{1}{2}\boldsymbol{\Sigma\Sigma}^{(i)}\right]^{-1}\left(\mathbf{1}_i+m\cdot\mathbf{S}^{(i)}\mathbf{k}_1^{(i)}\right) \tag{66}$$

with $\mathbf{k}_1^{(n)}=\mathbf{0}$.

We are left with the following proposition.

Proposition 5. *The equity value functions* $\mathbf{E}$ *for a given default vector* $\boldsymbol{y}_B$ *are*

$$\mathbf{E}(y)=\begin{cases}\underbrace{\mathbf{E}^{(n)}(y)}_{n\times 1}=\mathbf{GG}^{(n)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(n)}y\right)\cdot\mathbf{cc}^{(n)}+\mathbf{KK}^{(n)}\exp\left(\boldsymbol{\Gamma}^{(n)}y\right)\mathbf{c}^{(n)}+\mathbf{kk}_0^{(n)}+\mathbf{kk}_1^{(n)}\exp(y) & y\in I_n\\ \vdots & \vdots\\ \underbrace{\mathbf{E}^{(i)}(y)}_{i\times 1}=\mathbf{GG}^{(i)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{cc}^{(i)}+\mathbf{KK}^{(i)}\exp\left(\boldsymbol{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)}+\mathbf{kk}_0^{(i)}+\mathbf{kk}_1^{(i)}\exp(y) & y\in I_i\\ \vdots & \vdots\\ \underbrace{\mathbf{E}^{(1)}(y)}_{1\times 1}=\mathbf{GG}^{(1)}\cdot\exp\left(\boldsymbol{\Gamma\Gamma}^{(1)}y\right)\cdot\mathbf{cc}^{(1)}+\mathbf{KK}^{(1)}\exp\left(\boldsymbol{\Gamma}^{(1)}y\right)\mathbf{c}^{(1)}+\mathbf{kk}_0^{(1)}+\mathbf{kk}_1^{(1)}\exp(y) & y\in I_1\end{cases}$$

with the following boundary conditions to pin down the vector $\mathbf{cc}^{(i)}$*:*

$$\lim_{y\to\infty}\left|\underbrace{\mathbf{E}^{(n)}\left(y\right)\exp\left(-y\right)}_{n\times 1}\right| < \infty \tag{67}$$

$$\underbrace{\mathbf{E}^{(i+1)}\left(y_B\left(i+1\right)\right)}_{(i+1)\times 1} = \underbrace{\begin{bmatrix}\mathbf{E}^{(i)}\left(y_B\left(i+1\right)\right)\\ 0\end{bmatrix}}_{(i+1)\times 1} \tag{68}$$

$$\underbrace{\left(\mathbf{E}^{(i+1)}\right)'\left(y_B\left(i+1\right)\right)_{[1:i]}}_{i\times 1} = \underbrace{\left(\mathbf{E}^{(i)}\right)'\left(y_B\left(i+1\right)\right)}_{i\times 1} \tag{69}$$

$$\underbrace{\mathbf{E}^{(i)}\left(y_B\left(1\right)\right)}_{i\times 1} = 0 \tag{70}$$

where $\mathbf{x}_{[1:i]}$ *selects the first* i *rows of vector* $\mathbf{x}$*.*

Note first the dimensionalities: $\underbrace{\mathbf{\Gamma\Gamma}^{(i)}}_{2i\times 2i}, \underbrace{\mathbf{GG}^{(i)}}_{i\times 2i}$ and $\underbrace{\mathbf{\Gamma}^{(i)}}_{4i\times 4i}, \underbrace{\boldsymbol{G}^{(i)}}_{2i\times 4i}$. Note second the derivative of the equity value vector is

$$\underbrace{\left(\mathbf{E}^{(i)}\right)'(y)}_{i\times 1} = \mathbf{GG}^{(i)}\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{cc}^{(i)} + \mathbf{KK}^{(i)}\mathbf{\Gamma}^{(i)}\exp\left(\mathbf{\Gamma}^{(i)}y\right)\mathbf{c}^{(i)} + \mathbf{kk}_1^{(i)}\exp\left(y\right) \tag{71}$$

where we note that $\mathbf{\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma}^{(i)}$ and $\mathbf{\Gamma\Gamma}^{(i)}\cdot\exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right) = \exp\left(\mathbf{\Gamma\Gamma}^{(i)}y\right)\cdot\mathbf{\Gamma\Gamma}^{(i)}$as both are diagonal matrices (although this interchangeability only is important when $s = 1$ as it then helps collapse some equations).

The optimality conditions for bankruptcy boundaries $\{y_B\left(i\right)\}_i$ are given by

$$\left(\mathbf{E}^{(i)}\right)'\left(y_B\left(i\right)\right)_{[i]} = 0 \tag{72}$$

i.e., a smooth pasting condition at the boundaries at which default is declared.

B Holding Costs Microfoundation

Here, we introduce a simple framework based on collaterialized borrowing to result in holding costs that depend on current bond value.

Suppose that high types can borrow at discount rate r, but low types can only borrow at rate r if collateralized, and have to borrow at rate $r + \chi$ for all uncollateralized amounts. Suppose further that, independent of the purchasing history of the agent, there is a fixed outstanding loan L that results in constant negative consumption rL. If the agent gets hit by a liquidity shock, this negative consumption jumps to $(r + \chi)L$. Thus, the agent is exposed to an additional negative cash-flow of $\chi \cdot L$ due to the liquidity shock.

However, the agent is able to reduce this exposure to the additional spread by using the bond as collateral to raise $[1 - h(y)]P(y)$ where $h(y)$ is a haircut function and $P(y) = \frac{A(y)+B(y)}{2}$ is the midpoint price at a rate r. Thus, the cost from the spread that applied before is now reduced to $\chi(L - [1 - h(y)]P)$, and the ownership of the bond conveys a **marginal** value of the amount $\chi[1 - h(y)]P(y)$ until the time of sale. At the time of sale, the bond conveys a **marginal** value of $\frac{\chi\cdot B(y)}{r}$ on top of the sale value $B(y)$ as the agent does not leave the liquidity state and discounts negative/positive cash-flows forever at a rate r.

We further assume that there is a holding costs $N(y)$ that is unrelated to the loan portfolio or haircuts of the firm. This could be a cost that is related to the current price of the bond ($P(y)$), volatility of the bond price ($\sigma P'(y)$), or simply a fixed charge independent of the state of the firm.

Thus, we are facing the following situation in marginal terms. An agent in possession of the bond has a marginal valuation

$$r_H D_H(y) = c + \mathcal{L}D_H(y) + \xi_{HL}\left[D_L(y) - D_H(y)\right] \tag{73}$$

$$r_L D_L(y) = c + \chi\left[1 - h(y)\right]P(y) + \mathcal{L}D_L(y) + \lambda\left[B(y) + \frac{\chi \cdot B(y)}{r_L} - D_L(y)\right] \tag{74}$$

where we assume that flow-payments in the L state cannot be used to pay down the debt, only lumpsum payment can. Suppose that with probability β, the investor can make a take-it-or-leave-it offer to the dealer, and with probability $(1-\beta)$ the dealer can make the offer the the investor. The dealer's outside option is 0, and his valuation of the bond is is simply $D_H(y)$, the price at which he could sell on the secondary market.

If the dealer gets to make the offer, he will require a B s.t. $\left[B(y) + \frac{\chi \cdot B(y)}{r_L} - D_L(y)\right] = 0$, which gives

$$B_d(y) = \frac{r_L}{r_L + \chi} D_L(y) \tag{75}$$

If the investor gets to make the offer, he will require the highest B s.t. the dealer is willing to accept, which is given by

$$B_i(y) = D_H(y) \tag{76}$$

Thus, with probability β, surplus of $\left[\frac{r_L+\chi}{r_L} D_H(y) - D_L(y)\right]$ accrues to the investor, and with probability $(1-\beta)$, zero surplus accrues to the investor. Thus, we have

$$... + \lambda\beta\left[\frac{r_L + \chi}{r_L} D_H(y) - D_L(y)\right] \tag{77}$$

and we see that cash has a Lagrange multiplier $\frac{r_L+\chi}{r_L} > 1$ in the liquidity state. Further, note that we have

$$P(y) = \frac{A(y) + B(y)}{2} = \frac{D_H(y) + (1-\beta) D_H(y) + \beta \frac{r_L}{r_L+\chi} D_L(y)}{2} = \left(1 - \frac{\beta}{2}\right) D_H(y) + \frac{\beta}{2} B_d(y) \tag{78}$$

Multiply the D_L equation by $\frac{r_L}{r_L+\chi}$, and subtract $\xi_{HL}\frac{\chi}{r_L} D_H(y)$ from both sides of the $D_H(y)$ equation. Then, we rewrite to get

$$\left(r_H - \xi_{HL}\frac{\chi}{r_L}\right) D_H(y) = c + \mathcal{L}D_H(y) + \xi_{HL}\frac{r_L + \chi}{r_L}\left[\frac{r_L}{r_L + \chi} D_L(y) - D_H(y)\right] \tag{79}$$

$$r_L B_d(y) = \frac{r_L}{r_L + \chi}\left[c + \chi\left[1 - h(y)\right]P(y)\right] + \mathcal{L}B_d(y) + \lambda\beta\left[D_H(y) - B_d(y)\right] \tag{80}$$

Let $r_H = r_L + \xi_{HL}\frac{\chi}{r_L}$. Then, we have

$$r_L D_H(y) = c + \mathcal{L}D_H(y) + \xi_{HL}\frac{r_L + \chi}{r_L}\left[B_d(y) - D_H(y)\right] \tag{81}$$

$$r_L B_d(y) = \frac{r_L}{r_L + \chi}\left[c + \chi\left[1 - h(y)\right]P(y)\right] + \mathcal{L}B_d(y) + \lambda\beta\left[D_H(y) - B_d(y)\right] \tag{82}$$

Note that we can redefine the $H \to L$ jump term as $\hat{\xi}_{HL} \equiv \xi_{HL}\frac{r_L+\chi}{r_L}$ flow term as

$$\frac{r_L}{r_L + \chi}\left[c + \chi\left[1 - h(y)\right]P(y)\right] = c - \underbrace{\frac{\chi}{r_L + \chi}\left[c - r_L\left[1 - h(y)\right]P(y)\right]}_{\text{holding cost}}$$

and the second term can be interpreted as holding costs. And, given constant N, when

$$h(y) = \frac{\chi N + r_L N - c}{r_L P(y)} - \frac{\chi}{r_L}$$

we obtain the holding cost as $hc(y) = \chi(N - P(y))$. Letting $r = r_L$, we then recover the original pricing equations.

Given $N = 107$, $\chi_G = 12\%$ and $\chi_B = 17\%$ and other baseline calibrations, we find that model-implied haircuts range from 10% to 25% depending on ratings. They are broadly consistent with \citet{bis:10} which gives a comprehensive survey conducted by the BIS Committee on the Global Financial System (including both Europe and US) for June 2007 and June 2009. The model implied haircuts are 9% for Aaa/Aa bonds, 10% for A, 12\% for Baa, and 18% for Ba. In \citet{bis:10}, the average haircuts for non-rated counterparties are 6.7\% for Aaa/Aa rated bonds, 12\% for Baa, and 23\% for high yield bonds.

C Empirical Leverage Distribution across Ratings

[FIGURE 1 ABOUT HERE]

[FIGURE 2 ABOUT HERE]

[TABLE C ABOUT HERE]

Table 1: **Baseline Parameters used in calibration.** Unreported parameters are tax rate $\pi = 0.35$, and bond face value $p = 100$. Panel A reports pre-fixed parameters. Transition density $\zeta^{\mathbb{P}}$, jump risk premium $\exp(\kappa)$, risk price η, risk-free rate r, cash flow growth $\mu_{\mathbb{P}}$, primary bond market issuance cost κ, and inverse of debt maturity m are similar to the literature (e.g., Chen, Xu, and Yang (2012)). Systematic volatility σ_m is chosen to match equity volatility. Meeting intensity λ are set so that selling holdings takes one week (2.5 weeks) in normal (recession) period. The liquidity shock intensity ξ_B is set to match bond turnover in Bond-CDS sample in recession, and we set $\xi_G = \xi_B$. Investors' bargaining power β is from Feldhütter (2012). State- and type- dependent recovery rates α_l^s's are calculated using existing literature on credit risk models and observed bid-ask spreads of defaulted bonds. In Section **??**. Panel B reports four calibrated parameters. The idiosyncratic volatility σ_i is chosen to target the default probability of Baa firms. The holding cost parameters N is set to target the Baa state-G Bond-CDS spread, and χ_s's are chosen to target the investment grade bid-ask spreads.

Symbol	Description	State G	State B	Justification / Target
		A. Pre-fixed parameters		
$\zeta^{\mathbb{P}}$	Transition density	0.1	0.5	literature
$\exp(\kappa)$	Jump risk premium	2.0	0.5	literature
η	Risk price	0.165	0.255	literature
r	Risk free rate	0.05		literature
$\mu_{\mathbb{P}}$	Cash flow growth	0.045	0.015	literature
σ_m	Systematic vol	0.1	0.11	equity vol
ω	Primary market issuance cost	0.01		literature
m	Average maturity intensity	0.2		literature
λ	Meeting intensity	50	20	anecdotal evidence
ξ	Liquidity shock intensity	2		turnover in B, Bond-CDS sample
β	Investor's bargaining power	0.05		literature
α_H	Recovery rate of H type	58.71%	32.56%	literature
α_L	Recovery rate of L type	57.49%	30.50%	literature
		B. Calibrated parameters		
σ_i	Idiosyncratic vol	0.225		Baa default probability
N	Holding cost intercept	107		Baa Bond-CDS spread in G
χ	Holding cost slope	0.12	0.17	Investment bid-ask spread

Table 2: **Default probabilities and credit spreads across credit ratings.** Default probabilities are cumulative default probabilities over 1920-2011 from Moody's investors service (2012), and credit spreads are from FISD transaction data over 1994-2010. We report the time series mean, with the standard deviation (reported underneath) being calculated using Newey-West procedure with 15 lags. The standard deviation of default probabilities are calculated based on the sample post 1970's due to data availability issue. On model part, we first calculate the quasi market leverage for Compustat firms (excluding financial and utility firms) for each rating over 1994-2010 (exclude the crisis quarters 2008Q4 and 2009Q1), then match observed quasi market leverage by locating the corresponding cash flow level y. We calculate the average of model-implied credit spreads and Bond-CDS spreads across these firm-quarter observations. This procedure implies that our model-implied leverages exact match the empirical counterpart.

	Maturity = 5 years				Maturity = 10 years			
	Aaa/Aa	A	Baa	Ba	Aaa/Aa	A	Baa	Ba
	Panel A. Default probability (%)							
data	0.7	1.3	3.1	9.8	2.1	3.4	7.0	19.0
model	0.5	1.5	3.7	9.9	2.5	5.8	10.9	20.0
$\chi = 0$	0.4	1.0	2.8	8.2	2.0	4.7	9.3	17.7
	Panel B. Credit spreads (bps)							
	State G							
data	55.7	85.7	149	315	61.2	90.2	150	303
	(3.7)	(6.6)	(15.5)	(33.8)	(4.4)	(6.3)	(12.8)	(22.7)
model	72.9	103	170	341	108	193	305	490
$\chi = 0$	10.1	26.5	68.8	194	26.5	61.1	119	234
	State B							
data	107	171	275	542	106	159	262	454
	(5.8)	(10.5)	(23.9)	(29.8)	(6.7)	(13.8)	(29.3)	(44.4)
model	99	148	243	459	146	259	409	642
$\chi = 0$	12.9	33.8	84.0	221	29.4	70.0	133	255

Table 3: **Bond-CDS spreads and bid-ask spreads across credit ratings.** In Panel A, the sample to construct Bond-CDS spreads are firms with both 5-year and 10-year bonds, over the sample period from 2005 to 2012. We report the time series mean (excluding crisis period from October 2008 to March 2009), with the standard deviation (reported underneath) being calculated using Newey-West procedure with 15 lags. On the model side, we calculate the quasi market leverage for Compustat firms (excluding financial and utility firms) for each rating classes. We match the observed quasi market leverage by locating the corresponding cash flow level y, and calculate the time series average of model-implied credit spreads and Bond-CDS spreads across these firm-quarter observations (excluding crisis period from 2008Q3 to 2009Q1). The row of $\chi = 0$ gives the model implied moments when there is no liquidity frictions under our baseline parameters. In Panel B, the normal time bid-ask spread are taken from Edwards, Harris, and Piwowar (2007) for a median size trade. The recession time numbers are normal time numbers multiplied by the ratio of bid-ask spread implied by Roll's measure of illiquidity (following Bao, Pan, and Wang (2011)) in recession time to normal time. The model counterpart is computed for a bond with time to maturity of 8 years, which is the mean time-to-maturity of frequently traded bonds (where we can compute a Roll (1984) measure) in the TRACE sample.

	Panel A. Bond-CDS spreads (bps)							
	Maturity = 5 years				Maturity = 10 years			
	Aaa/Aa	A	Baa	Ba	Aaa/Aa	A	Baa	Ba
	State G							
data	27.7	44.4	74.6	104	23.2	37.2	58.5	67.8
	(6.6)	(5.8)	(8.7)	(11.2)	(9.9)	(6.1)	(9.0)	(16.1)
model	55.5	62.1	78.6	99.6	71.7	108	161	201
	State B							
data	76.0	125	182	227	72.2	104	162	191
	(5.1)	(2.1)	(18.0)	(39.2)	(3.4)	(6.1)	(22.0)	(36.5)
model	74.5	90.6	116	160	105	154	229	311

	Panel B. Bid-Ask spreads (bps)					
	State G			State B		
	Superior	Investment	Junk	Superior	Investment	Junk
data	40	50	70	77	125	218
model	38	50	91	104	127	282

Table 4: **Structural Liquidity-Default Decomposition for 5-Year Bonds Across Ratings.** For each firm-quarter observation, we locate the corresponding cash flow level y that delivers the observed quasi market leverage in Compustat (excluding financial and utility firms). We perform the structural liquidity-default decomposition for a 5-year bond following the procedure discussed in the text, for each rating and each aggregate states.

Rating	State	Credit Spread	Default-Liquidity Decomposition *Pure Def*	*Pure Liq*	$Liq \to Def$	$Def \to Liq$
		Panel I: Explaining Credit Spread Levels				
Aaa/Aa	G (bps)	72.9	9.7	45.9	2.8	14.4
	(%)	100	13	63	4	20
	B (bps)	99.1	13.8	61.1	4.1	20.1
	(%)	100	14	62	4	20
A	G (bps)	103	25.8	45.9	8.9	22.0
	(%)	100	25	45	9	21
	B (bps)	148	36.1	61.1	12.2	38.3
	(%)	100	24	41	8	26
Baa	G (bps)	170	65.5	45.9	20.5	37.9
	(%)	100	39	27	12	22
	B (bps)	243	86.9	61.1	26.7	68.5
	(%)	100	36	25	11	28
Ba	G (bps)	341	178	45.9	45.3	71.9
	(%)	100	52	13	13	21
	B (bps)	459	215	61	57.5	124.7
	(%)	100	47	13	13	27
		Panel II: Explaining Credit Spread Changes				
Aaa/Aa	$G \to B$ (bps)	26.2	4.1	15.2	1.3	5.6
	(%)	100	16	58	5	22
A	$G \to B$ (bps)	45.0	10.3	15.2	3.3	16.3
	(%)	100	23	34	7	36
Baa	$G \to B$ (bps)	73.5	21.5	15.2	6.2	30.6
	(%)	100	29	21	8	42
Ba	$G \to B$ (bps)	118	37.6	15.2	12.2	52.8
	(%)	100	32	13	10	45

Table 5: **Effect of Liquidity Provision Policy on 5-Year Bonds Across Ratings.** We consider a policy experiment that improves the liquidity condition (χ, λ) in the B state to be as good as G state. We compute the credit spread under the policy for both G and B state, and perform the structural liquidity-default decomposition to examine the channels that are responsible for the reduced borrowing cost.

Rating	State	Credit Spread (rf)		Contribution of Each Component		
		w/o. policy	w. policy	pure LIQ (%)	LIQ→DEF (%)	DEF→LIQ (%)
Aaa/Aa	G	55.7	38.9	58	7	35
	B	82.5	40.1	56	3	40
A	G	85.7	53.7	31	37	32
	B	134	61.9	33	22	45
Baa	G	149	87.9	16	38	46
	B	237	111	19	39	42
Ba	G	315	186	8	51	42
	B	481	244	10	45	45

Table 6: Matrix & Vector Dimensions.

Debt Parameters			Equity Parameters		
Symbol	Interpretation	Dimension	Symbol	Interpretation	Dimension
$\mathbf{D}^{(i)}(y)$	Debt Value Function	$2i \times 1$	$\mathbf{E}^{(i)}(y)$	Equity Value Function	$i \times 1$
$\boldsymbol{\mu}^{(i)}$	(Log)-Drifts	$2i \times 2i$	$\boldsymbol{\mu\mu}^{(i)}$	(Log-)Drifts	$i \times i$
$\boldsymbol{\Sigma}^{(i)}$	Volatilities	$2i \times 2i$	$\boldsymbol{\Sigma\Sigma}^{(i)}$	Volatilities	$i \times i$
$\mathbf{R}^{(i)}$	Discount rates and maturity	$2i \times 2i$	$\mathbf{RR}^{(i)}$	Discount rates	$i \times i$
$\boldsymbol{\chi}^{(i)}$	Holding cost slopes	$2i \times 2i$	c	Coupon	1×1
$\mathbf{N}^{(i)}$	Holding cost intercepts	$2i \times 1$	π	Tax rate	1×1
$\mathbf{Q}^{(i)}$	Transition to cont. states	$2i \times 2i$	$\mathbf{QQ}^{(i)}$	Transition to cont. states	$i \times i$
$\tilde{\mathbf{Q}}^{(i)}$	Transition to default states	$2i \times 2(n-i)$	$\mathbf{AA}^{(i)}$	Matrix to be decomposed	$2i \times 2i$
$\mathbf{v}^{(i)}$	Vector of recovery values	$2(n-i) \times 1$	$\boldsymbol{\Gamma\Gamma}^{(i)}$	Diag matrix of eigenvalues	$2i \times 2i$
$\mathbf{W}^{(i)}$	Mid-point weighting matrix	$2i \times 2i$	$\mathbf{GG}^{(i)}$	Matrix of eigenvectors	$i \times 2i$
$\boldsymbol{\Gamma}^{(i)}$	Diag matrix of eigenvalues	$4i \times 4i$	$\mathbf{kk}_0^{(i)}, \mathbf{kk}_1^{(i)}$	Coeff. of particular sol.	$i \times 1$
$\mathbf{G}^{(i)}$	Matrix of eigenvectors	$2i \times 4i$	$\mathbf{S}^{(i)}$	Issuance matrix	$i \times 2i$
$\mathbf{k}_0^{(i)}, \mathbf{k}_1^{(i)}$	Coeff. of particular sol.	$2i \times 1$	$\mathbf{KK}^{(i)}$	Coeff. of particular sol.	$i \times 4i$
$\mathbf{c}^{(i)}$	Vector of constants	$4i \times 1$	$\mathbf{cc}^{(i)}$	Vector of constants	$2i \times 1$

Figure 1: **Empirical Distribution of Market Leverage for Compustat Firms by Aggregate State and Rating classes.** We compute quasi-market leverage for each firm-quarter observation in the Compustat database from 1997-2012. The B state is defined as quarters for which at least two months are classified as NBER recession month. The remaining quarters are G state. We drop financial and utility firms in our sample. We also exclude firms with zero leverage.

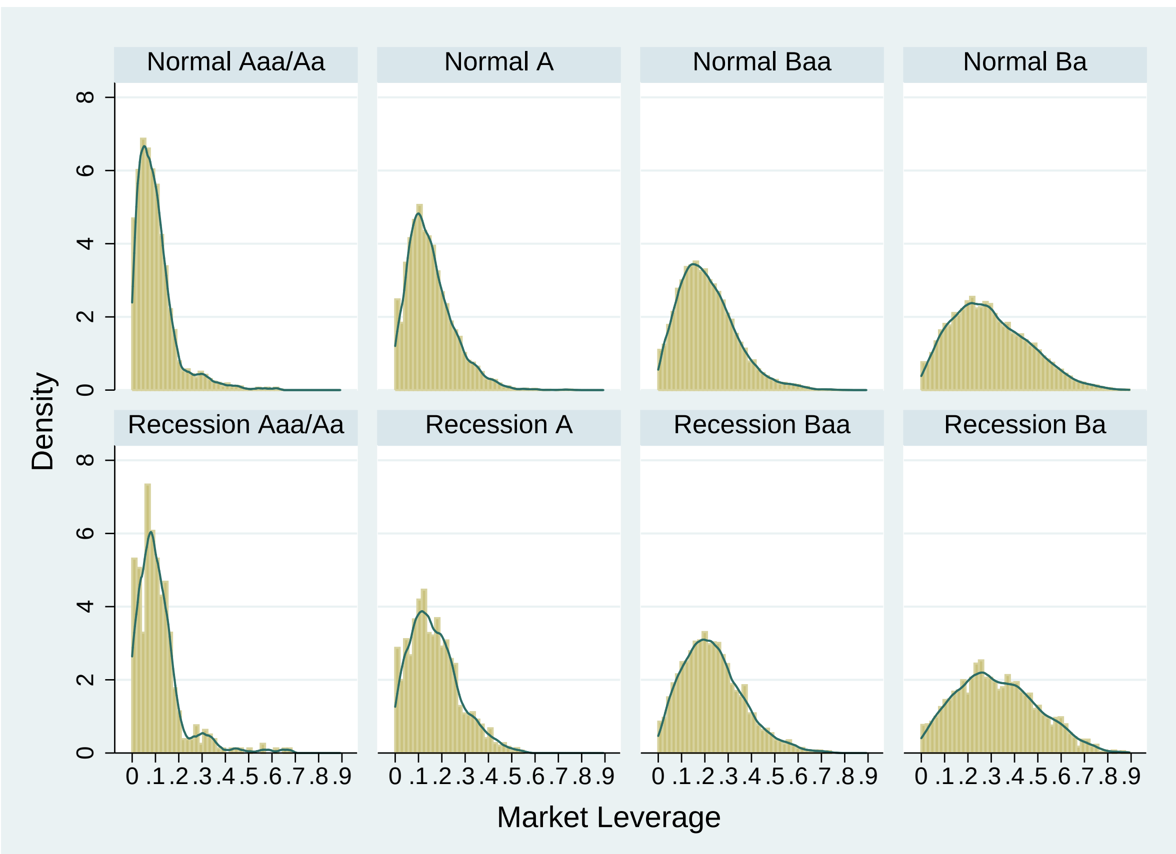

Figure 2: Model Implied Nonlinearity between Market Leverage, Default Rates and Total Credit Spread